Friday Fish

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A Categorical Prologue

Sites and sheaves

Def. Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. A <u>sieve</u> S on A is a right ideal in $\mathrm{Hom}_{\mathcal{C}}(\cdot, A)$, i.e., a family of arrows with target A s.t. $f \in S \Rightarrow fg \in S$ if $t(g) = s(f)$.

If $f : B \to A$, the pullback sieve on B is

$f^* S := \{ g : t(g) = B, fg \in S \}$.

Def. A <u>Grothendieck topology</u> J on $\mathcal{C}$ is a function that assigns to each $A \in \mathcal{C}$ a family of sieves $J(A)$, called covering sieves, s.t.

(1) $t_A := \{ f : t(f) = A \} = \mathrm{Hom}_{\mathcal{C}}(\cdot, A) \in J(A)$,

(2) (stability) if $f : B \to A$, then $S \in J(A) \Longrightarrow f^* S \in J(B)$,

(3) (transitivity) if $S \in J(A)$ and R is a sieve on A s.t. $f^* R \in J(s(f))$ for all $f \in S$, then $R \in J(A)$.

A <u>site</u> is a pair $(\mathcal{C}, J)$, where J is a Grothendieck topology.

<u>Ex</u>. Let $(\mathcal{C}, J)$ be a site. Then:

(i) the transitivity axiom is equivalent to: if $S \in J(A)$ and there is $R_f \in J(s(f))$ for each $f \in S$, then $\{fg : f \in S, g \in R_f\} \in J(A)$.

(ii) $R, S \in J(A) \implies R \cap S \in J(A)$.

Maybe the language of bases is more familiar.

<u>Def</u>. A <u>basis</u> for a Grothendieck topology on $\mathcal{C}$ is a function K that assigns to each $A \in \mathcal{C}$ a family of arrows $K(A)$, called covering families, s.t.

(1) if $f : B \xrightarrow{\cong} A$, then $\{f\} \in K(A)$,

(2) (stability) if $f : B \to A$, then $\{f_i : A_i \to A\}_i \in K(A)$ $\implies$ $A_i \times_A B$ exist and $\{A_i \times_A B \to B\}_i \in K(B)$,

(3) (transitivity) if $\{f_i : A_i \to A\}_i \in K(A)$ and for each i there is $\{g_{ij} : B_{ij} \to A_i\}_j \in K(A_i)$, then $\{f_i g_{ij} : B_{ij} \to A\}_{i,j} \in K(A)$.

<u>Prop</u>. A basis K generates a Grothendieck topology J by declaring that $S \in J(A)$ iff there is $R \in K(A)$ s.t. $R \subseteq S$.

<u>Ex</u>. (1) Small site of $X \in \mathrm{Top}$.

Objects = open sets, arrows = inclusions.

Basis: $\{U_i \subseteq U\}_i \in K(U)$ if $U = \bigcup_i U_i$.

The covering sieve generated by $\{U_i \subseteq U\}_i$ is the maximal refinement of $\{U_i\}_i$: the set of open sets V s.t. $V \subseteq U_i$ for some i.

(2) Large site of X: ~~as~~ category $\mathrm{Top}_{/X}$,

~~Sections~~ Let $f: Y \to X$, then $\{f_i : Y_i \to Y\}_i \in K(f)$ iff f_i open embeddings s.t. $Y = \bigcup_i f_i(Y_i)$.

If $X = *$, then $\mathrm{Top}_{/X} = \mathrm{Top}$ (~~as sites~~).

$\mathcal{C}_{/A} = t_A$, with morphisms commuting triangles
$$\begin{tikzcd} B \arrow[r] \arrow[dr] & C \arrow[d] \\ & A \end{tikzcd}$$

(3) Large site of $M \in \mathrm{Mfd}$. Same as $\mathrm{Top}_{/X}$, $\mathrm{Mfd}_{/M}$.

Only possible issue: pullbacks. But open emb. are submersions, so in this case it works.

(4) $\mathbb{R}^\circ$ full subcategory of Mfd whose objects are $\{\mathbb{R}^0, \mathbb{R}^1, \mathbb{R}^2, \dots\}$. The Euclidean site $\mathrm{Euc} \subseteq \mathrm{Mfd}$ is the full subcategory of M obtained from $\mathbb{R}^\circ$ by taking ~~of~~ open subsets and disjoint unions, with the usual open cover topology.

<u>Def</u>. A <u>presheaf</u> on a category $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to \mathrm{Set}$. Notation: if $f : B \to A$ and $x \in F(A)$, then $x|_B := F(f)(x)$.

If $\mathcal{C}$ is a site with a basis K, then F is a <u>sheaf</u> if for all $\{f_i : A_i \to A\}_i \in K(A)$ and $x_i \in F(A_i)$ s.t. $x_i|_{A_{ij}} = x_j|_{A_{ij}}$, where $A_{ij} = A_i \times_A A_j$, there is a unique $x \in F(A)$ s.t. $x|_{A_i} = x_i$.

<u>Ex</u>. (1) A sheaf on the small site of $X \in \mathrm{Top}$ is a sheaf in the ~~ty~~ typical sense.

~~Fibered Ca~~

Fibered categories

<u>Def.</u> A category $\mathcal{D}$ is <u>fibered in groupoids</u> over $\mathcal{C}$ if there is a functor $\pi : \mathcal{D} \to \mathcal{C}$ s.t.

(1) if $f \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ and $C \in \mathcal{D}$, with $\pi(C) = A$, then there is $\tilde{f} \in \mathrm{Hom}_{\mathcal{D}}(D, C)$ s.t. $\pi(\tilde{f}) = f$,

(2) if we have

$$\begin{tikzcd} C \arrow[dr, "\tilde{g}"] & \\ & A \\ B \arrow[ur, "\tilde{f}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \pi(C) \arrow[r, "\pi(\tilde{g})"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & \pi(A) \\ \pi(B) \arrow[ur, "\pi(\tilde{f})"'] & \end{tikzcd}$$

then there is a unique $\tilde{h} \in \mathrm{Hom}_{\mathcal{D}}(C, B)$ s.t. $\tilde{f}\tilde{h} = \tilde{g}$ and $\pi(\tilde{h}) = h$.

We define the <u>fiber</u> π_A over $A \in \mathcal{C}$ as the subcategory of $\mathcal{D}$ whose objects are $\{B \in \mathcal{D} : \pi(B) = A\}$ and whose arrows are $\{\tilde{f} \in \mathrm{Hom}_{\mathcal{D}}(C, B) : \pi(B) = \pi(C) = A, \pi(\tilde{f}) = 1_A\}$.

<u>Prop.</u> $\forall A \in \mathcal{C}$, π_A is a groupoid (all morphisms are inv.)

<u>Proof.</u> $B, C \in \pi_A$, $f : B \to C$ with $\pi(f) = 1_A$.

$$\begin{tikzcd} C \arrow[dr, "1_C"] \arrow[dd, dotted, "g"'] & \\ & C \\ B \arrow[ur, "f"'] & \end{tikzcd} \quad \begin{tikzcd} A \arrow[dr, "1_A"] \arrow[dd, "1_A"'] & \\ & A \\ A \arrow[ur, "1_A"'] & \end{tikzcd} , \text{ and } \begin{tikzcd} B \arrow[dr, "1_B"] \arrow[dd, dotted, "h"'] & \\ & B \\ C \arrow[ur, "g"'] & \end{tikzcd} \quad \begin{tikzcd} A \arrow[dr, "1_A"] \arrow[dd, "1_A"'] & \\ & A \\ A \arrow[ur, "1_A"'] & \end{tikzcd}$$

Then $fg = 1_C$, and $fgh = 1_C h = h = f 1_B = f$. #

<u>Def.</u> A fibered category $\pi : \mathcal{D} \to \mathcal{C}$ is called discretely fibered if π_A is a set (groupoid with no nonidentity arrows), i.e., if for all $B, C \in \pi_A$ and $\tilde{f} : C \to B$ with $\pi(\tilde{f}) = 1_A$ we have that $B = C$ and $\tilde{f} = 1_B$.

<u>Ex.</u> (1) Representable fibered categories.

$A \in \mathcal{C}$, consider $\pi : \mathcal{C}_{/A} \to \mathcal{C}$ given by

$\pi(B \to A) = B$, $\pi\left(\begin{tikzcd}[column sep=small] B \arrow[rr] \arrow[dr] & & C \arrow[dl] \\ & A & \end{tikzcd}\right) = B \to C$

~~If $f : B \to C$ and $g : C \to A$, so that $\pi(g) = C$, then gf~~

Check the conditions:

- $f : C \to B$ and $g : B \to A$, so that $\pi(g) = B$,

then $gf : C \to A$, so $\pi(fg) = C$ and

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "gf"'] & & B \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

is a morphism in $\mathcal{C}_{/A}$ whose image under π is f.

•

$$\begin{tikzcd} C \arrow[dr, "f"] \arrow[drr, bend left] & & \\ & B \arrow[r] & A \\ D \arrow[ur, "g"'] \arrow[urr, bend right] & & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[dr, "f"] \arrow[dd, "h"'] & \\ & B \\ D \arrow[ur, "g"'] & \end{tikzcd}$$

Notice that $\begin{tikzcd} C \arrow[rr, "h"] \arrow[dr] \arrow[rr, phantom, shift right=3, "\circlearrowleft"] & & D \arrow[dl] \\ & A & \end{tikzcd}$, since ~~B~~ $(D \to A) \circ h =$

$= (B \to A) \circ gh = (B \to A) \circ f = C \to A$,

hence, h is the unique lift of h.

Actually, $\mathcal{C}_{/A} \to \mathcal{C}$ is discretely fibered: if $f, g \in \pi_A$,

$f : B \to A$, $g : C \to A$, and there is h s.t.

$\begin{tikzcd} C \arrow[rr, "h"] \arrow[dr, "g"'] \arrow[rr, phantom, shift right=3, "\circlearrowleft"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$ and $\pi(h) = h = 1_A$, then $B = C = A$.

(2) If $f : B \to A \in \mathcal{C}_{/A}$, we can form the fibered category $\pi : \mathcal{C}_{/B} \to \mathcal{C}_{/A}$ defined by $\pi(C \to B) = f \circ (C \to B)$,

$\pi\left(\begin{tikzcd} D \arrow[rr] \arrow[dr] & & C \arrow[dl] \\ & B & \end{tikzcd}\right) = \begin{tikzcd} D \arrow[rr] \arrow[dr, "f \circ (D \to B)"'] & & C \arrow[dl, "f \circ (C \to B)"] \\ & A & \end{tikzcd}$. Also discretely fibered.

(3) Actual stacky stuff.

G topological group, $\mathcal{C} = \mathrm{Prin}_G$ with objects principal G-bundles and morphisms G-equiv. maps.

$\pi : \mathrm{Prin}_G \to \mathrm{Top}$ given by $\pi(P \to X) = X$ and

$$\pi\left(\begin{tikzcd} P_1 \arrow[r, "\tilde{f}"] \arrow[d] & P_2 \arrow[d] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd}\right) = f$$

Check properties:

- $$\begin{tikzcd} & P \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$ Then $f^*P = P \times_Y X \to X$ is a principal G-bundle.

The map $pr_1 : P \times_Y X \to P$ is G-equiv:

$pr_1((p,x)g) = pr_1(pg, x) = pg = pr_1(p,x)\, g$.

- $$\begin{tikzcd} R \arrow[dr, "\tilde{g}"] & \\ & Q \\ P \arrow[ur, "\tilde{f}"'] & \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[d, "h"'] \arrow[dr, "g"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

If $r \in R_z$, take any $p \in P_{h(z)}$ ~~s.t.~~ and let $h \in G$ be s.t. $\tilde{g}(r) = \tilde{f}(p)h$. Set $\tilde{h}(r) := ph$.

Exercise: well-defined, continuous, unique.

In particular, $\pi_X = \mathrm{Prin}_G(X)$ is a groupoid.

2-categories

<u>Def.</u> A (strict) <u>2-category</u> $\mathcal{C}$ consists of

(i) a class of objects,

(ii) a category $\mathcal{C}(A,B)$ for ~~each~~ objects A, B,

1-morphisms:	objects of $\mathcal{C}(A,B)$ $\rightarrow$
2-morphisms:	morphisms of $\mathcal{C}(A,B)$ $\Rightarrow$
vertical comp. $*$:	comp in $\mathcal{C}(A,B)$

(iii) $A \in \mathcal{C}$, there is $1_A \in \mathcal{C}(A,A)$ and a 2-morphism $1_{1_A} : 1_A \Rightarrow 1_A$,

(iv) $A, B, C \in \mathcal{C}$ there is a functor $\mathcal{C}(B,A) \times \mathcal{C}(C,B) \longrightarrow \mathcal{C}(C,A)$, called horizontal comp. $\circ$,

such that

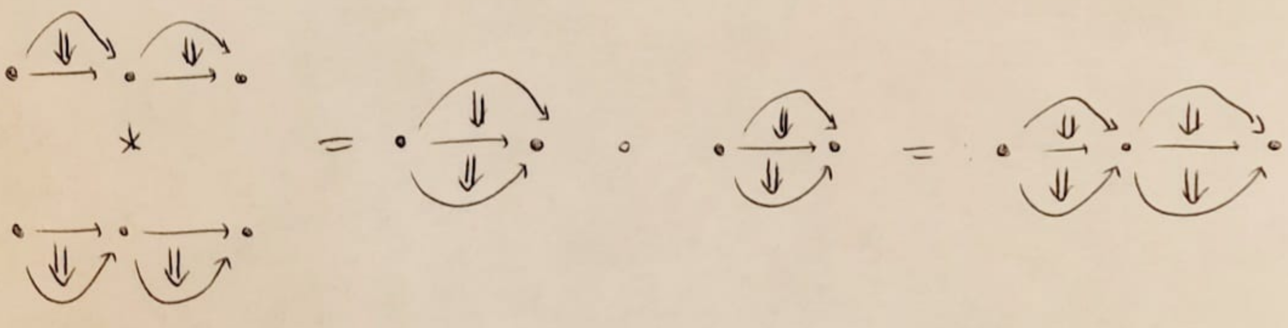
(1) Horizontal composition is associative and unital

$$\left(\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd} \circ \begin{tikzcd} B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & C \end{tikzcd} \right) \circ \begin{tikzcd} C \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & D \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd} \circ \left(\begin{tikzcd} B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & C \end{tikzcd} \circ \begin{tikzcd} C \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & D \end{tikzcd} \right) =:$$

$$= \begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & C \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow 1_{1_B}"] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow 1_{1_A}"] & A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

(2) Vertical and horizontal comp. can be interchanged.



<u>Ex</u>. Cat is a 2-category (small categories).

$\mathrm{Cat}(\mathcal{C}, \mathcal{D}) = [\mathcal{C}, \mathcal{D}]$, with natural transformations.

<u>Def</u>. A <u>lax 2-functor</u> $F : \mathcal{C}^{op} \to \mathrm{Cat}$ consists of the following:

(i) $F(A) \in \mathrm{Cat}$ for $A \in \mathcal{C}$,

(ii) $F(f) : F(A) \to F(B)$ functor for $f : B \to A$,

(iii) $\varepsilon_A : F(1_A) \xrightarrow{\simeq} 1_{F(A)}$ for $A \in \mathcal{C}$

(iv) if $C \xrightarrow{g} B \xrightarrow{f} A$, $\alpha_{g,f} : F(g)F(f) \simeq F(fg)$.

S.t.

(1) $B \xrightarrow{f} A$ and $C \in F(A)$, then

$$\begin{tikzcd}
F(1_B)\,F(f)(C) \arrow[r, "\varepsilon_B(F(f)(C))"] \arrow[d, "\alpha_{1_B,f}(C)"'] \arrow[dr, phantom, "\circlearrowleft"] & 1_{F(B)}(F(f)(C)) \arrow[dl, equal] \\
F(f)(C) &
\end{tikzcd}$$

$$\begin{tikzcd}
F(f)\,F(1_A)\,(C) \arrow[rr, "F(f)(\varepsilon_A(C))"] \arrow[d, leftrightarrow, "\alpha_{f,1_A}(C)"'] \arrow[drr, phantom, "\circlearrowright"] & & F(f)\,(1_{F(A)}(C)) \\
F(f)\,(C) \arrow[urr, equal] & &
\end{tikzcd}$$

(2) $D \xrightarrow{h} C \xrightarrow{g} B \xrightarrow{f} A$ and $E \in F(A)$, then ?

$$\begin{tikzcd}
F(h)\,F(g)\,F(f)(E) \arrow[r, "\alpha_{h,g}(F(f)(E))"] \arrow[d, "F(h)(\alpha_{g,f}(E))"'] \arrow[dr, phantom, "\circlearrowright"] & F(gh)\,F(f)(E) \arrow[d, "\alpha_{gh,f}(E)"] \\
F(h)\,F(fg)(E) \arrow[r, "\alpha_{h,fg}(E)"'] & F(fgh)(E)
\end{tikzcd}$$

We can think of fibered categories as "presheaf of groupoids".

Def. A cleavage of a fibered category $\pi : D \to C$ is a collection K of arrows of D such that for every arrow $f : B \to A$ in $\mathcal{C}$ and each object $C \in \pi_A$, there is a unique $\tilde{f} \in K$ such that $\pi(\tilde{f}) = f$. We denote $t(\tilde{f}) := f^*C$.

Prop. $\pi : \mathcal{D} \to \mathcal{E}$ fibered category, then π defines a lax 2-functor $F : \mathcal{E}^{op} \to \mathrm{Grpd}$ as follows:

- $F(A) := \pi_A$,
- $f : B \to A$, then $F(f) : F(A) \to F(B)$ is defined

by $F(f)(C) := f^*C$, for $C \in F(A)$, and if $g : D \to C$ for $C, D \in \pi_A$ and $\pi(g) = 1_A$, then we ~~denote~~ ~~by~~ let $F(f)(g) := f^*g$.

$$
\begin{tikzcd}
f^*D \arrow[r] \arrow[d, "f^*g"'] & D \arrow[d, "g"] \\
f^*C \arrow[r, "\tilde{f}"'] \arrow[ur, dashed] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "1_B"'] & A \arrow[d, "1_A"] \\
B \arrow[r, "f"'] \arrow[ur, "f"] & A
\end{tikzcd}
$$

<u>Sketch of proof.</u> $C \xrightarrow{g} B \xrightarrow{f} A$, $D \in F(A)$

$$
\begin{tikzcd}
g^*(f^*D) \arrow[dr] \arrow[d, dashed, shift right, "\alpha_{g,f}(D)"'] & \\
(fg)^*D \arrow[r] \arrow[u, dashed, shift right, "\alpha^{-1}_{g,f}(D)"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[dr, "fg"] \arrow[d, shift right, "1_C"'] & \\
C \arrow[r, "fg"'] \arrow[u, shift right, "1_C"'] & A
\end{tikzcd}
$$

#

Ex. (1) $A \in \mathcal{C}$ and representable fibered category $\pi : \mathcal{C}_{/A} \to \mathcal{C}$. The presheaf that it defines is $F : \mathcal{C}^{op} \to \mathrm{Set}$, with ~~$F(A) = \pi_A = \mathrm{Hom}_{\mathcal{C}}(\cdot,$~~

$F(B) = \pi_B = \mathrm{Hom}_{\mathcal{C}}(B, A)$, with only identity morphisms.

Pullback morphisms: if $f : C \to B$ and $g \in \pi_B$, then $gf \in \pi_C$ and

$$\pi\left(\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "gf"'] & B \arrow[d, "g"] \\ & A \end{tikzcd}\right) = f,$$

so $F(f) : F(B) \to F(C)$ is exactly given by $F(f)(g) = gf$. Hence $F = \mathrm{Hom}_{\mathcal{C}}(\cdot, A)$ is a representable presheaf.

(2) $\pi : \mathrm{Prin}_G \to \mathrm{Top}$. The usual pullback defines a cleavage. But if $Z \xrightarrow{g} Y \xrightarrow{f} X$ and $P \in \mathrm{Prin}_G(X)$, then $g^* f^* P = (P \times_X Y) \times_Y Z \neq (gf)^* P = P \times_X Z$, but they are canonically isomorphic:

$$\begin{aligned} (P \times_X Y) \times_Y \times Z &\longrightarrow P \times_X Z \\ (p, y, z) &\longmapsto (p, z) \end{aligned}$$