ERRATA

The following is a list of errata for our book 'Simplicial and dendroidal homotopy theory'. We thank Kensuke Arakawa, Miguel Barata, Thomas Blom, Vladimir Hinich, Francesca Pratali, and Sven van Nigtevecht for pointing out some of the inaccuracies addressed below.

p.16, Definition 1.20. In the definition of a tree, the function $O\colon V \to E$ should be injective. (Every vertex is supposed to have precisely one output edge, of course.)

p.107, Example 3.20(c). In the sentence following the display, e should be x.

p.254, proof of Theorem 6.51. In the last sentence of the first paragraph, it is not automatic that $k(A, X)$ is a Kan complex. Indeed, this is generally only true if A is a *normal* dendroidal set. However, this case suffices for the proof, since the saturated class of normal monomorphisms is generated by normal monos between normal dendroidal sets.

p.354, Definition 9.1. There is a typo here: $n > 0$ should read $n \geq 0$.

p.358, Theorem 9.9. Although not explicitly stated, it is useful to note that the model structure of the theorem is cofibrantly generated.

p.486, paragraph before Lemma 12.6. Here it is claimed that for a Reedy cofibrant dendroidal space A, the cosimplicial object $A \boxtimes \Delta[\bullet]$ is a cosimplicial resolution of A, with a reference to Example 11.15. However, that example concerns the projective model structure, not the Reedy one. Nonetheless, the same argument given there will apply here. Indeed, to check that $A \boxtimes \Delta[\bullet]$ is a Reedy cofibrant cosimplicial object it suffices to show that for each $n \geq 0$ and each Reedy cofibration of dendroidal spaces $i\colon U \to V$, the map

$$U \boxtimes \Delta[n] \cup V \boxtimes \partial\Delta[n] \to V \boxtimes \Delta[n],$$

is again a Reedy cofibration of dendroidal spaces. One can take i to be a generating Reedy cofibration of the form

$$T \boxtimes \partial\Delta[m] \cup \partial T \boxtimes \Delta[m] \to T \boxtimes \Delta[m],$$

in which case the map above becomes

$$T \boxtimes (\partial\Delta[m] \times \Delta[n] \cup \Delta[m] \times \partial\Delta[n]) \cup \partial T \boxtimes (\Delta[m] \times \Delta[n]) \to T \boxtimes (\Delta[m] \times \Delta[n]).$$

Clearly this is again a Reedy cofibration.

p.508. The proof of Corollary 12.42 has a gap. We are checking associativity of the derived tensor product $\otimes^{\mathbb{L}}$ on the homotopy category $\mathrm{Ho}(\mathbf{dSpaces}_{RSC})$. For dendroidal spaces X and Y, we have defined $X \otimes^{\mathbb{L}} Y$ by taking projectively cofibrant replacements of X and Y and then taking the tensor product of those. To prove associativity of the derived tensor product, it will suffice to check that for projectively cofibrant X, Y, and Z and a projectively cofibrant replacement $W \to X \otimes Y$, the composite 'associator map'

$$W \otimes Z \to (X \otimes Y) \otimes Z \to X \otimes Y \otimes Z$$

is a weak equivalence. Note that the current proof of Corollary 12.42 does not take the cofibrant replacement W into account and only deals with the second map. By the usual skeletal induction we may reduce to the case where X and Y are represented by trees S and T respectively. (If desired, one can reduce to the case where Z is a tree as well.) It remains to argue the following:

Lemma 1. *The map*

$$W \otimes Z \to (S \otimes T) \otimes Z$$

is a weak equivalence in $\mathbf{dSpaces}_{RS}$.

The proof of the lemma relies on the observation that certain colimit diagrams are actually *homotopy* colimits. To be precise, we call a diagram $f\colon I \to \mathcal{E}$ in a cofibrantly generated model category $\mathcal{E}$ a *homotopy colimit diagram* if the natural map $\mathrm{hocolim}_I f \to \mathrm{colim}_I f$ is a weak equivalence. (Of course the term $\mathrm{hocolim}_I f$ is only well-defined up to weak equivalence, but for the definition it does not matter.) To state our preparatory lemma, write $S \otimes T$ as a union $\cup_i A_i$ of shuffles A_i with $1 \leq i \leq n$. Write $P(n)$ for the poset of nonempty subsets of $\{1, \ldots, n\}$, ordered by reverse inclusion. If we set $A_U := \cap_{i \in U} A_U$, then we can write

$$S \otimes T \cong \mathrm{colim}_{U \in P(n)} A_U.$$

Lemma 2. *The diagrams*

$$U \mapsto A_U \qquad \textit{and} \qquad U \mapsto A_U \otimes Z$$

are homotopy colimit diagrams in $\mathbf{dSpaces}_R$ *(or, equivalently, in* $\mathbf{dSpaces}_P$*).*

Proof. Since the identity functor gives a Quillen equivalence between the projective and the Reedy model structures on **dSpaces**, the notion of homotopy colimit diagram is indeed the same with respect to both model structures. Interpret $P(n)$ as a Reedy category in which every morphism is positive. For a general model category $\mathcal{E}$, we observe that the projective and Reedy model structures agree on $\mathcal{E}^{P(n)}$. Thus, a diagram $f\colon P(n) \to \mathcal{E}$ is projectively cofibrant precisely if for every $U \in P(n)$, the latching map

$$\mathrm{colim}_{U \subsetneq V} f(V) \to f(U)$$

is a cofibration in $\mathcal{E}$. In the specific case where $\mathcal{E} = \mathbf{dSpaces}_R$ and $f(U) = A_U$, this becomes the inclusion

$$\bigcup_{U \subsetneq V} A_V \subseteq A_U.$$

This is a normal monomorphism of dendroidal sets (since A_U is representable, hence normal) and therefore a Reedy cofibration of dendroidal spaces. It follows that the diagram is projectively cofibrant and its colimit is 'the' homotopy colimit.

For the second diagram, first note that $- \otimes Z$ preserves intersections between shuffles, so that $A_U \otimes Z = \cap_{i \in U}(A_i \otimes Z)$. It follows that in this case the latching map may be identified with the inclusion of the subobject

$$\bigcup_{U \subsetneq V} (A_V \otimes Z) \subseteq A_U \otimes Z.$$

Again, this is a normal monomorphism and the proof is complete. □

Proof of Lemma 1. It follows from Lemma 12.45 that the tensor product with Z defines a left Quillen functor

$$- \otimes Z\colon \mathbf{dSpaces}_{PS} \to \mathbf{dSpaces}_{RS}.$$

In particular, it preserves weak equivalences between projectively cofibrant objects and it suffices to prove the lemma for a single choice of cofibrant replacement $W \to S \otimes T$. Take a projectively cofibrant replacement Y of the diagram $U \mapsto A_U$ in the functor category $\mathbf{dSpaces}_P^{P(n)}$. The colimit is a left Quillen functor, so that

$\varinjlim_{P(n)} Y$ is a projectively cofibrant dendroidal space. Moreover, it is a model for the homotopy colimit of $U \mapsto A_U$, so that the map $\varinjlim_{P(n)} Y \to A$ is a projective weak equivalence by Lemma 2. Thus, we may take $W = \varinjlim_{P(n)} Y$ as our cofibrant replacement.

Since $- \otimes Z$ is left Quillen it preserves homotopy colimits, so that $W \otimes Z$ is the homotopy colimit of the diagram

$$P(n) \to \mathbf{dSpaces}_{RS} \colon U \mapsto A_U \otimes Z.$$

From the second part of Lemma 2 we conclude that the map

$$W \otimes Z \to \mathrm{colim}_{U \in P(n)} A_U \otimes Z \cong (S \otimes T) \otimes Z$$

is a weak equivalence. □

Finally, for a different approach to constructing a symmetric monoidal structure on the homotopy category of dendroidal spaces we refer to the appendix of 'On the equivalence of Lurie's ∞-operads and dendroidal ∞-operads' by Hinich–Moerdijk.

p.516. The proof of Theorem 12.60 and its Corollary 12.61 only apply to open reduced dendroidal spaces, rather than closed ones. The reason is that the 'reduction functor' introduced above the theorem is only well-defined on open trees. In particular, Theorem 12.62 remains valid.

p.528. In the proof of Proposition 13.7 we implicitly use the following observation:

Lemma 3. *Let $f_! \colon \mathcal{E} \rightleftarrows \mathcal{F} \colon f^*$ be a Quillen equivalence between model categories. Suppose that $\mathcal{E}_\lambda$ and $\mathcal{F}_{\lambda'}$ are left Bousfield localizations of $\mathcal{E}$ and $\mathcal{F}$ respectively. If f^* preserves and detects local objects (meaning X is λ'-local if and only if f^*X is λ-local), then the pair $f_! \colon \mathcal{E}_\lambda \rightleftarrows \mathcal{F}_{\lambda'} \colon f^*$ is a Quillen equivalence as well.*

Proof. If f^* preserves local objects, then its left adjoint $f_!$ sends λ-equivalences to λ'-equivalences and is therefore also left Quillen for the localized model structures. Consider the following diagram of right adjoints:

$$\begin{array}{ccc} \mathrm{Ho}(\mathcal{E}) & \xleftarrow{\mathbf{R}f^*} & \mathrm{Ho}(\mathcal{F}) \\ \uparrow & & \uparrow \\ \mathrm{Ho}(\mathcal{E}_\lambda) & \xleftarrow{\mathbf{R}f^*} & \mathrm{Ho}(\mathcal{F}_{\lambda'}). \end{array}$$

The vertical arrows are fully faithful (being the inclusions of the full subcategories on local objects) and the upper horizontal arrow is an equivalence of categories by assumption. Hence the bottom arrow is fully faithful as well. It is essentially surjective by the assumption that f^* detects local objects. We conclude that the bottom arrow is an equivalence of categories. □

p.530, right above Lemma 13.9. We are considering a projectively fibrant dendroidal space B and a Reedy fibrant replacement $f \colon B \to B'$ of it, inducing a Quillen equivalence

$$\mathbf{dSpaces}_P/B \xrightarrow{f_!} \mathbf{dSpaces}_R/B'.$$

We claim that pushing forward the covariant localization of the projective model structure $\mathbf{dSets}_P/B$ to the model category $\mathbf{dSets}_R/B'$ gives the covariant localization $(\mathbf{dSpaces}_R/B')_{\mathrm{cov}}$. To see this, one needs that every covariant localizing morphism $\ell[T] \to T \to B'$ in $\mathbf{dSpaces}_R/B'$ can be lifted, up to weak equivalence, to a covariant weak equivalence in $\mathbf{dSpaces}_P/B$. Since $B \to B'$ is a projective weak equivalence, any morphism $T \to B'$ can be lifted to a morphism $T \to B$ *up* *to homotopy*; this suffices to find a covariant localizing morphism $\ell[T] \to T \to B$ in $\mathbf{dSpaces}_P/B$ lifting the previous one up to homotopy.

p.574. Lemma 14.24: The proof is an inductive argument on the generating sparse cofibrations. However, it doesn't explicitly address the base case of the induction, which is the evident observation that for the dendroidal space $X = C_k \boxtimes \Delta[n]$ the map $X \to N\tau(X)$ is an isomorphism.