

Title: Lectures on the de Rham complex

Abstract:

My goal in these lectures is to discuss recent advances on de Rham cohomology in positive or mixed characteristic, due to Bhatt-Scholze, Bhatt-Lurie, Drinfeld, and Petrov. However, de Rham cohomology has a long and rich history. So in order to put these new results into perspective, in the first part I will briefly recall some of its milestones.

A. *A brief historical survey.* (Lecture 1)

Highlights: the Poincaré lemma and the de Rham theorem, the analytic de Rham complex and Hodge theory, the algebraic de Rham complex, de Rham cohomology in characteristic zero, the Cartier isomorphism, crystalline cohomology.

B. *New results around Deligne-Illusie (after Drinfeld, Bhatt-Lurie, and Petrov).* (Lectures 2, 3)

If Y is a smooth scheme over a perfect field k of characteristic $p > 0$, the main result of [DI87] asserts that if Y is liftable to a smooth scheme $X/W_2(k)$, then the canonical truncation in degree $< p$ of its de Rham complex $\Omega_{Y/k}^\bullet$ decomposes into the sum of its shifted cohomology sheaves. A new theory, *prismatization*, due independently to Drinfeld and Bhatt-Lurie, shows that the datum of a lifting X as above makes this decomposition fit into a canonical $\mathbb{Z}/p\mathbb{Z}$ -grading on $\Omega_{Y/k}^\bullet$, and produces an additional operator, called the *Sen operator*, whose nilpotent part is rather mysterious.

A corollary of the main result of [DI87] is that if Y/k is liftable to $W_2(k)$, and proper and smooth of dimension $\leq p$, the Hodge to de Rham spectral sequence of Y/k degenerates at E_1 . A question left open since then was: does there exist a proper, smooth scheme of dimension $p + 1$ over k , liftable to $W_2(k)$, whose Hodge to de Rham spectral sequence does not degenerate at E_1 . An affirmative answer has just been given by A. Petrov. I will explain some of the salient points of his construction, which partly relies on the results of Bhatt-Lurie-Drinfeld just mentioned, a new insight into the derived functor of S^p , and delicate refinements of famous results of Cline-Parshall-Scott-van der Kallen [CPSvdK] on the cohomology of reductive groups in positive characteristic.

I would also like to discuss another recent result of Petrov, of positive nature, this time, namely that quasi-F-split smooth projective varieties in positive characteristic have decomposable de Rham complexes.

Finally, time permitting, I will sketch complements and open problems.

[DI87] Deligne, Pierre; Illusie, Luc. *Relèvements modulo p^2 et décomposition du complexe de de Rham.* Invent. Math. 89 (1987), no. 2, 247–270.

[CPSvdK] Cline, Edward; Parshall, Brian; Scott, Leonard; van der Kallen, Wilberd. *Rational and generic cohomology*. *Invent. Math.* 39 (1977), no. 2, 143–163.