Wall-crossing for Virasoro constraints (joint with M. Moreira and W. Lim)

Arkadij Bojko

ETH Zürich, Department of Mathematics

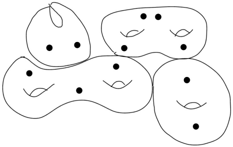
November 22, 2022

Structure of the talk.

1. History and background for Virasoro constraints
2. History and Background for wall-crossing and vertex algebras
3. Geometric construction of Joyce's vertex algebras
4. How the two intertwine
5. Applications
   - 5.1 Quivers
   - 5.2 Curves and Surfaces
   - 5.3 What next?
6. Idea of the proof.

Witten's conjecture

1. Moduli space of algebraic pointed curves $\overline{\mathcal{M}}_{g,n}$ parametrizing $(C, x_1, \ldots, x_n)$:

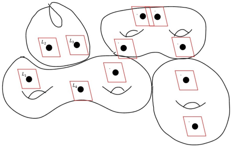


,

Figure: $(C, x_1, \ldots, x_n)$

   where we label the points by $x_1, \ldots, x_n$.

2. There are line bundles $L_i \to \overline{\mathcal{M}}_{g,n}$ which at each point of it (as in 2) are given by $T^*_C|_{x_i}$. Denote the powers of the first Chern classes by $\tau_d = c_1(L_i)^d$:



,

Figure: $L_i|_{(C,x_1,\cdots,x_n)} = T^*_C|_{x_i}$

KdV hierarchy

1. *Gustav de Vries* and *Diederik Johannes Korteweg* studied the differential equation describing waves on shallow water in a canal:

$$\frac{\partial}{\partial t}\Phi(x,t) = \Phi(x,t)\frac{\partial}{\partial x}\Phi(x,t) + \frac{\partial^3}{\partial x^3}\Phi(t,x)\,.$$

2. For a function $\Phi(\vec{t};x)$, there is a generalization of the above called *KdV hierarchy*:

$$\begin{aligned}
\frac{\partial}{\partial t_1}\Phi =& \Phi\frac{\partial}{\partial x}\Phi + \frac{\partial^3}{\partial x^3}\Phi \\
\frac{\partial}{\partial t_2}\Phi =& \frac{1}{2}\Phi^2\frac{\partial^3}{\partial x^3}\Phi + \frac{1}{6}\frac{\partial}{\partial x}\Phi\frac{\partial^2}{\partial x^2}\Phi - \frac{1}{200}\frac{\partial^4}{\partial x^4}\Phi \\
&\vdots \\
\frac{\partial}{\partial t_k}\Phi =& \frac{\partial}{\partial x}P_k(\Phi, \frac{\partial}{\partial x}\Phi, \frac{\partial^2}{\partial x^2}\Phi, \cdots)\,.
\end{aligned}$$

3. Amazingly, by comparing two different approaches to 2-dimensional quantum gravity, Witten expected that the integrals

$$\langle \tau_0^{a_0}\tau_1^{a_1}\cdots\tau_k^{a_k}\rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \tau_0^{a_0}\tau_1^{a_1}\cdots\tau_k^{a_k}$$

once fit into the generating series

$$F(\vec{t},x) = \sum_{g\geq 0} F_g(x,\vec{t})\lambda^{2g-2} = \sum_{\vec{a},g,n} \langle \tau_0^{a_0}\tau_1^{a_1}\cdots\tau_k^{a_k}\rangle_{g,n}\frac{x^{a_0}}{a_0!}\cdots\frac{t_k^{a_k}}{a_k!}\lambda^{2g-2}\,.$$

Witten's conjecture II

1. ... satisfy the KdV hierarchy of PDE's.
2. Set $\Phi = \frac{\partial}{\partial x^2} F(x, \vec{t})$, then

$$\frac{\partial}{\partial t_k}\Phi = \frac{\partial}{\partial x} P_k\left(\Phi, \frac{\partial}{\partial x}\Phi, \frac{\partial^2}{\partial x^2}\Phi, \cdots\right)$$

   for all $k \geq 1$.
3. Additionally, it satisfies the *string equation*:

$$\frac{\partial}{\partial x} F = \frac{x^2}{2} + \sum_i t_{i+1} \frac{\partial}{\partial t_i} F\,.$$

4. It was proved famously by Kontsevich (92') and later by Mirzakhani (07') who both won Fields Medals in part for this work.

Virasoro constraints

1. Dijkgraaf, Verlinde and Verlinde (90') defined a sequence of second order differential operators L_k on $\mathbb{C}[\![x, t_1, \cdots, t_n]\!]$, such that the τ-function
$$\tau = \exp\left[\frac{1}{2\lambda^2} F(x, \vec{t})\right]$$
satisfies
$$L_k \tau = 0\,.$$
2. T. Eguchi, K. Hori and C.-S. Xiong (97') extended these operators for $\overline{\mathcal{M}}_{g,n}(X, \beta)$ which parametrizes $(C, f, x_1, \ldots, x_n)$ for a map $f : C \to X$ such that $f([C]) = \beta \in H_2(X)$. They conjectured that similar vanishings hold in this case.
3. This was proved by Okounkov[1]–Pandharipande (03') for curves $X = C$ and by Givental (01') and Teleman (12') for toric X.
4. Fix a basis $B = \{v\} \subset H^*(X)$ with 1 denoting the generator of $H^0(X)$ being one of its elements, then $F_g(x, \vec{t})$ is replaced by
$$F_g^X = \sum_{\substack{\vec{a}, \vec{k} n, \\ v_i \in B}} \left\langle \left(\tau_{k_1}(v_1)\right)^{a_1} \left(\tau_{k_2}(v_2)\right)^{a_2} \cdots \left(\tau_{k_l}(v_l)\right)^{a_l} \right\rangle_{g,n} \frac{\left(t_{k_1}^{v_1}\right)^{a_1}}{a_1!} \cdots \frac{\left(t_{k_l}^{v_l}\right)^{a_l}}{a_l!}$$

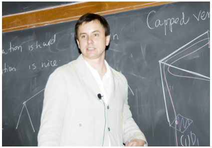
[1] Another Fields Medalist who received it partly due to his work on Virasoro constraints.

Virasoro constraints II

1. where

$$\left\langle \left(\tau_{k_1}(v_1)\right)^{a_1}\left(\tau_{k_2}(v_2)\right)^{a_2}\cdots\left(\tau_{k_l}(v_l)\right)^{a_l}\right\rangle_{g,n}$$
$$= \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \left(\tau_{k_1}(v_1)\right)^{a_1}\left(\tau_{k_2}(v_2)\right)^{a_2}\cdots\left(\tau_{k_l}(v_l)\right)^{a_l}.$$

The *descendent classes* $\tau_{k_i}(v_i) = \tau^{k_i}\mathrm{ev}_i^*(v_i)$ are defined in terms of $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$ mapping each $(C, x_1, \ldots, x_n, f)$ to $f(x_i)$.



Virasoro constraints for sheaves and pairs

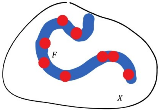
1. When moving on to the sheaf side of the story, consider either a moduli space M parametrizing sheaves $[F]$ or P parametrizing morphisms $\mathcal{O}_X \to F$ on X with the universal sheaf or pair

$$\mathbb{G} \quad \text{on} \quad X \times M\,, \qquad \text{respectively} \qquad \mathcal{O}_{X\times P} \to \mathbb{G} \quad \text{on} \quad X \times P\,,$$

   the descendent classes are replaced by

$$\begin{aligned} ch_i^H(v) &= \pi_{2\,,*}\Big(\pi_1^*(v)\mathrm{ch}_{i+\dim(X)-p}(\mathbb{G})\Big)\,,\\ \mathrm{ch}_i(v) &= \mathrm{ch}^H_{i+\lfloor \frac{p-q}{2}\rfloor}(v)\,, \qquad v \in H^{p,q}(X)\,, \end{aligned}$$

   where $X \xleftarrow{\pi_1} X \times M \xrightarrow{\pi_2} M$ are the projections.
2. The main result of Moreira–Oblomkov–Okounkov–Pandharipande(20') is transporting Virasoro constraints from GW theory to PT stable pairs moduli space P parametrizing $\mathcal{O}_X \xrightarrow{s} F$, where $\dim\mathrm{coker}(s) = 0$ and F is a pure sheaf in some cases (toric X, stationary descendents)



Virasoro constraints for sheaves: set up

1. To formulate Virasoro constraints in a nice way, it is useful to extend the moduli space M to include all sheaves and even more their complexes

$$\cdots \to F_a \to F_{a+1} \to \cdots F_{b-1} \to F_b \to \cdots$$

   The result is denoted by $\mathcal{M}_X$ and by Gross(19') has a completely explicit homology

$$H_*(\mathcal{M}_X) = \mathbb{C}[\![t_i^v : v \in B, i \geq 1]\!] .$$

2. The universal sheaf is also extended to $\mathcal{G}$ on $X \times \mathcal{M}_X$, where the same definition leads to classes $\mathrm{ch}_i(v)$ acting on the homology as a derivative:

$$\mathrm{ch}_k(v) \cap (-) = \frac{\partial}{\partial t_k^v} .$$

3. Using the virtual fundamental class $[M]^{\mathrm{vir}}$, because M is not smooth in general, define

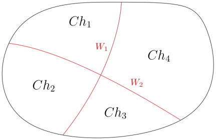
$$\langle \mathrm{ch}_{k_1}(v_1) \cdots \mathrm{ch}_{k_n}(v_n) \rangle_M = \int_{[M]^{\mathrm{vir}}} \mathrm{ch}_{k_1}(v_1) \cdots \mathrm{ch}_{k_n}(v_n)$$

Virasoro constraints for sheaves: formula

1. Using the previously introduced notation, it becomes clear that
$$\iota_*[M]^{\mathrm{vir}} = \sum_{\vec{k}, v_i \in B} \left\langle \mathrm{ch}_1(1)^{a_{1,1}} \cdots \mathrm{ch}_1(v)^{a_{1,v}} \cdots \mathrm{ch}_k(w)^{a_{k,w}} \right\rangle_M \frac{\left(t_1^1\right)^{a_{1,1}}}{a_{1,1}!} \cdots \frac{\left(t_k^w\right)^{a_{k,w}}}{a_{k,w}!}$$
replaces F^X in the GW potential.
2. Under the **Assumption A** that $H^{p,q}(X) = 0$ unless $|p-q| \leq 1$ one may now define the Virasoro operators L_k acting on $H_*(\mathcal{M}_X)$ by $L_k = T_k + R_k$, where
$$T_k = \sum_{\substack{i+j=k \\ v \in B}} (-1)^{\dim X - q} i! j! \mathrm{ch}_i^H(v) \mathrm{ch}_j^H(\bar{v} \cdot \mathrm{td}(X)) \cap ,$$
$$R_k = \sum_{\substack{j \geq 1 \\ v \in B}} \left(j + \left\lfloor \frac{p-q}{2} \right\rfloor\right)_{(k+1)} t_{j-k}^v \frac{\partial}{\partial t_j^v} ,$$
and $(a)_{(b)} = a(a-1)\cdots(a-b+1)$.
3. Fix α the K-theory class of sheaves in M. Because of some intertwining with GW theory, one additionally needs
$$S_k = -\frac{(k+1)!}{\mathrm{rk}(\alpha)} \mathrm{ch}_{k+1}^H(\mathrm{pt}) \circ R_{-1}$$
for the Virasoro constraints to hold.
4. **Claim:** In many cases the Virasoro constraints
$$(T_k + R_k + S_k)\iota_*[M]^{\mathrm{vir}} = 0$$
hold.

Wall-crossing

1. The idea of wall-crossing is simple. Changing some parameter in a parameter space Stab, one assigns to each point an invariant that jumps on some real codimension 1 walls:



   This appears already in Donaldson theory, where the parameter is a metric and the invariants are Donaldson invariants counting ASD instantons.

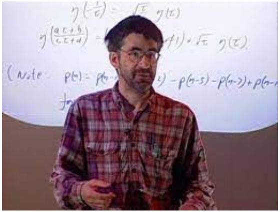
2. Joyce (03'-04') formulated a general framework for abelian categories that was later extended by Joyce–Song and Kontsevich–Soibelman to Calabi–Yau 3-categories and Donaldson–Thomas invariants.

3. Work in the abelian category of sheaves $\mathrm{Coh}(X)$ with a family of stability conditions on $\mathcal{A}$ denoted by W. For example, it could be a set of ample line-bundles H with the associated μ_H-stability $\mu_H(E) = \left(\mathrm{ch}_1(E)H^{d-1}\right)/\left(\mathrm{ch}_0(E)H^d\right)$ and E being (semi-)stable if for each subobject $E' \subset E$

$$\mu_H(E') \underset{(-)}{<} \mu_H(E)$$

Wall-crossing and vertex algebras

1. The wall-crossing invariants denoted by δ^H_α then give a (motivic) count of μ_H-semistable invariants in class α. Changing H to H' leads to *wall-crossing formulae* which express $\delta^{H'}_\alpha$ in terms of combinations of $\delta^H_{\alpha_i}$ such that
$$\alpha = \alpha_1 + \cdots + \alpha_k\,.$$
The expression contains Lie-brackets $[\delta_\alpha, \delta_\beta]$ coming from *Hall-algebras*.
2. The virtual fundamental classes $[M^H_\alpha]^{\mathrm{vir}}$ of μ_H-semistable sheaves in class α are not motivic. To remedy this, Gross–Joyce–Tanaka introduced wall-crossing in terms of vertex algebras.
3. Vertex algebras were introduce by Borcherds (86') to study infinite dimensional Lie algebras and later prove the Monstrous moonshine conjectures. He received his fields medal in the same year as M. Kontsevich in part for this definition.

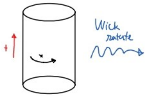


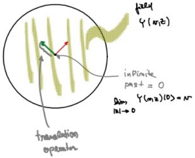
Vertex algebras

1. A vertex algebra is the data of a $\mathbb{Z}$-graded vector space V_* over $\mathbb{C}$ together with
   1.1 a *vacuum vector* $|0\rangle \in V_0$,
   1.2 a linear operator $T\colon V_* \to V_{*+2}$ called the *translation operator*,
   1.3 and a *state-field correspondence* which is a degree 0 linear map

$$Y\colon V_* \longrightarrow \mathrm{End}(V_*)[\![z, z^{-1}]\!]\,,$$

   denoted by $Y(a,z) \sum_{n\in} a_{(n)} z^{-n-1}$, where $\deg(z) = -2$.
2. These need to satisfy

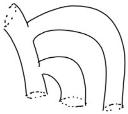




- $e^{zT}|0\rangle = |0\rangle$, $Y(|0\rangle, z) = \mathrm{id}$,
  – vacuum invariance
- $Y(v,z)w = e^{zT}\, Y(w,z)v$

– skew-symmetry

- $(z_1-z_2)^N\, Y(v,z_1)\, Y(w,z_2)$
  $= (z_1-z_2)^N\, Y(Y(v,z_1)w, z_2)$



Locality

Sketch of the geometric construction of vertex algebras?

1. Joyce (17') constructed vertex algebras on the homology $V_* = H_{*+\mathrm{vdim}_{\mathbb{C}}}(\mathcal{M}_X)$, which requires three ingredients

   1.1 The inclusion $p \xrightarrow{0} \mathcal{M}_X$ gives $|0\rangle = 0_*(p)$.

   1.2 There is an action of $B\mathbb{G}_m$ (the classifying stack of line bundles) on $\mathcal{M}_X$ which in terms of families can be rephrased as tensoring the universal sheaf $\mathcal{G}$ on $X \times \mathcal{M}_X$ by line bundles $\mathcal{L}$ on $\mathcal{M}_X$. Thus we get an action

   $$\Phi_* : \mathbb{C}[\![t]\!] \boxtimes H_*(\mathcal{M}_X) \to H_*(\mathcal{M}_X)$$

   with $T(-) = \Phi_*(t \boxtimes (-))$.

   1.3 Finally, the global $\mathrm{RHom}_{\mathcal{M}_X \times \mathcal{M}_X}(\mathcal{G}_{1,2}, \mathcal{G}_{1,3}) = \Theta$ is the last piece necessary to write down $Y(v, z)$, where $(-)_{i,j}$ denote pullbacks to the i, j-terms in $X \times \mathcal{M}_X \times \mathcal{M}_X$.

2. The quotient of $\mathcal{M}_X$ by $B\mathbb{G}_m$ exists in some setting and is denoted by $\mathcal{M}_X^{\mathrm{rig}}$. As one would expect, there is roughly the correspondence

   $$K_* = V_{*+2}/TV_* = H_{*+\mathrm{vdim}_{\mathbb{C}}}(\mathcal{M}_X^{\mathrm{rig}}).$$

3. Now there are two roles that K_* plays:

   3.1 There might not exist a map $\iota : M_\alpha^\sigma \to \mathcal{M}_X$, but there is $\iota' : M_\alpha^\sigma \to \mathcal{M}_X^{\mathrm{rig}}$ giving classes $\iota'_*[M_\alpha^\sigma]^{\mathrm{vir}} \in K_0$.

   3.2 K_* has the structure of a Lie algebra given by

   $$[\overline{v}, \overline{w}] = \overline{v_0 w}, \quad \forall v, w \in V_*.$$

Conformal element

1. A conformal element $\omega \in V_*$ is required to give a field $Y(\omega, z) = \sum_{k\in\mathbb{Z}} L_k z^{-k-2}$ with
$$\left[L_n, L_m\right] = (n-m)L_{m+n} + \frac{n^3-n}{12}\delta_{n+m,0}\cdot C\,,$$
2. The vertex algebra structure on V_* can be described (see Gross(19') + BML(22')) as a tensor product of its bosonic lattice part V_*^+ and fermionic lattice part V_*^- related to whether t_k^v generator satisfies $v \in H_\pm(X)$.
3. After choosing an isotropic subspace decomposition $I \oplus \bar{I} = H^-(X)$, there exist natural conformal elements
$$\omega_+ \in V_*^+\,, \qquad \omega_- \in V_*^-\,, \qquad \omega = \omega_+ + \omega_-$$
4. After introducing a new natural basis $t_k^{v,H}$ by
$$\mathrm{ch}_k^H(v) \cap (-) = \sum_{w\in B}\int_X v\cdot w\frac{\partial}{\partial t_k^{w,H}}\,,$$
it takes the natural looking form
$$\omega = \frac{1}{2}\sum_{v\in B} t_k^{v,H} t_k^{\hat{v},H}$$
with $\hat{v} \in \hat{B}$ denoting the dual basis with respect to the supersymmetrization of the following *holomorphic pairing*[2]
$$\chi^H(v,w) = (-1)^p\int_X v\cdot w\cdot \mathrm{td}(X)\,, \qquad v\in H^{p,q}(X)\,.$$

[2]Here one needs to be careful about non-degeneracy of the pairing, which we do not necessarily have. This can be fixed by working with pairs of perfect complexes instead.

Primary states

1. Borcherds also gave a definition of primary states $v \in P \subset V_2$ as satisfying the equations $L_k v = \delta_{k,0} v$ and showed that they form a Lie subalgebra $\check{P} \subset K_0^H$ (here H denotes a shifted grading on K like above).
2. We show that there is an operator

$$[-,\omega] = \sum_{n \geq -1} \frac{(-1)^n}{(n+1)!} T^{n+1} \circ L_n : K_0^H \to V_2^H$$

such that for *lattice vertex algebras* and their state v being primary is equivalent to the condition

$$[\bar{v}, \omega] = 0\,.$$

3. More importantly, there is now a map

$$[-,-] : K_* \times V_* \to V_*\,, \qquad \check{P} \times P \to P\,,$$

which relates to whether the moduli spaces admit a lift $\iota : M_\alpha^\sigma \to \mathcal{M}_X$ or not.

Main Theorem

Theorem

If ι' admits a lift $\iota : M_\alpha^\sigma \to \mathcal{M}_X$, the condition that $[M_\alpha^\sigma]^{vir}$ satisfies Virasoro constraints is equivalent to $\iota'_*[M_\alpha^\sigma]^{vir} \in K_0$ being a primary state with respect to the ω given above. I.e. $\iota'_*[M_\alpha^\sigma]^{vir} \in \check{P}$. In all other cases, we define Virasoro constraints by the virtual fundamental class being primary.

1. Since the wall-crossing formulae often take the form

$$[M_\alpha^{\sigma'}]^{\mathrm{vir}} = \sum_{\underline{\alpha} \vdash \alpha} U(\underline{\alpha}, \sigma, \sigma')\Big[[M_{\alpha_1}^\sigma]^{\mathrm{in}}, \big[[M_{\alpha_2}^\sigma]^{\mathrm{in}}, \ldots, \big[[M_{\alpha_l}^\sigma]^{\mathrm{in}}, [M_{\alpha_0}^\sigma]^{\mathrm{vir}}\big]\ldots\big]\Big]$$

   with $M_\alpha^{\sigma'}, M_{\alpha_0}^\sigma$ admitting lifts to $\mathcal{M}_X$ and $[M_{\alpha_1}^\sigma]^{\mathrm{in}}$ being homology classes essentially defined to satisfy the wall-crossing formulae[3], we see that Virasoro constraints are preserved by wall-crossing. I.e. if we know them for classes on the RHS, we know them for $M_\alpha^{\sigma'}$.
2. This allows us to prove them in multiple cases.

[3] and satisfying all the necessary comtatibilities and unieqeneness.

Rank reduction

Theorem (B.–Moreira–Lim(22'))

Let X be a curve or a surface with $H^1(S) = H^{2,0}(S) = 0$, then $[M_\alpha^H]^{in}$ satisfy Virasoro constraints whenever $rk(\alpha) > 0$.

1. This proves the Conjecture of van Bree (21').

Conjecture (B.–Moreira–Lim + idea of the proof)

The classes $[M_\alpha^H]^{in}$ for a surface satisfying the above conditions satisfy Virasoro constraints for any α.

2. The proof goes by induction on rank (in a suitable sense) starting
   2.1 from rank 1 case for a surface S, which is $[\mathrm{Hilb}^n(S)]$ proved by Moreira (21')
   2.2 from rank 0 case, which is $[M_{np}]^{\mathrm{in}}$ with an almost trivial computation.
3. In each inductive step, we have the formula

$$[P_\alpha]^{\mathrm{vir}} = \sum_{\underline{\alpha} \vdash \alpha} U(\underline{\alpha}) \big[[M_{\alpha_1}]^{\mathrm{in}}, \big[[M_{\alpha_2}]^{\mathrm{in}}, \dots, \big[[M_{\alpha_l}]^{\mathrm{in}}, [P_{\alpha_0}]^{\mathrm{vir}}\big] \dots \big]\big]$$

   for $\mathrm{rk}(\alpha) > \mathrm{rk}(\alpha_i)$ and $[M_{\alpha_i}]^{\mathrm{in}}$, $[P_{\alpha_0}]^{\mathrm{vir}}$ Virasoro constraints already known. I am neglecting to write H here repeatedly to improve the presentation.
4. Define $\Pi : \mathcal{P}_X \to \mathcal{M}_X$ the projection from the stack of pairs on X to sheaves and set

$$\Omega_\alpha^H = \Pi_*\big([P_\alpha]^{\mathrm{vir}} \cap c_{\mathrm{rk}}(\mathbb{T}_{\mathcal{P}_X/\mathcal{M}_X})\big)$$

   then Virasoro constraints are compatible with this operation and

$$\Omega_\alpha^H = [M_\alpha]^{\mathrm{in}} + \sum_{\underline{\alpha} \vdash \alpha} U'(\underline{\alpha}) \big[[M_{\alpha_1}]^{\mathrm{in}}, \big[[M_{\alpha_2}]^{\mathrm{in}}, \dots, \big[[M_{\alpha_l}]^{\mathrm{in}}, [M_{\alpha_0}]^{\mathrm{in}}\big] \dots \big]\big].$$

   with $\mathrm{rk}(\alpha) > \mathrm{rk}(\alpha_i)$. Reorganizing this proves the induction step.

Future

1. Get rid of the condition on S.
2. Quivers.
3. Calabi–Yau fourfolds.
4. Fano 3-folds?
5. Equivariant Virasoro constraints.