Concrete Abstract Computability Theory

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Mathematical Institute talk
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Computable functions

Computable *partial* functions $\mathbb{N} \rightharpoonup \mathbb{N}$.

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Need a notion of algorithm:

- imperative: Turing machines, register machines, . . . , Python;
- declarative: recursive functions, λ-calculus, . . . , Haskell.

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- declarative: recursive functions, λ-calculus, . . . , Haskell.

Two important observations:

- all these models yield the same notion of computable function $\mathbb{N} \rightharpoonup \mathbb{N}$;
- the number of computable functions is *countable*.

Leveling the playing field

Write φ_n for the computable partial function $\mathbb{N} \rightharpoonup \mathbb{N}$ given by the n-th algorithm (coding).

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Application function

We define a partial function $\mathbb{N} \times \mathbb{N} \rightharpoonup \mathbb{N}$ by:

$$n \cdot m \simeq \varphi_n(m).$$

Currying

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In general: a function $f : \mathbb{N}^k \rightharpoonup \mathbb{N}$ is computable iff there exists an $n \in \mathbb{N}$ such that

$$(\cdots((n \cdot m_1) \cdot m_2) \cdots) \cdot m_k \simeq f(m_1, \ldots, m_k).$$

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Write LHS as: $nm_1m_2 \cdots m_k$.

Combinatory completeness

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Theorem

For every term $t(x_1, \ldots, x_k)$, there exists an $n \in \mathbb{N}$ such that

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$$n m_1 \cdots m_k \simeq t(m_1, \ldots, m_k).$$

A *partial combinatory algebra* (PCA) is a set A equipped with an application function $A \times A \rightharpoonup A$ that has this property.

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The Scott topology

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$$U_p = \{A \subseteq \mathbb{N} \mid p \subseteq A\}$$

where $p \subseteq \mathbb{N}$ is *finite*.

Under $\mathcal{P}\omega \cong \{0,1\}^{\mathbb{N}}$, this is the product topology, where

$$\mathcal{O}(\{0,1\}) = \{\emptyset, \{1\}, \{0,1\}\}.$$

Scott-continuous functions

Observation

A function $F\colon \mathcal{P}\omega \to \mathcal{P}\omega$ is Scott-continuous iff

$$F(B) = \bigcup_{p \subseteq B \text{ finite}} F(p),$$

for all $B \subseteq \mathbb{N}$.

Scott-continuous functions

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for all $B \subseteq \mathbb{N}$.

In particular:

- a Scott-continuous function is order-preserving;
- there are $|\mathcal{P}\omega|$ Scott-continuous functions.

Coding continuous functions I

A Scott-continuous function F is determined by the set of all pairs (p, n), where $p \subseteq \mathbb{N}$ finite and $n \in F(p)$.

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Coding of pairs

Define the bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}, (m, n) \mapsto \langle m, n \rangle$ by:

$$\langle m, n \rangle = \frac{1}{2}(m + n)(m + n + 1) + n.$$

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Coding of finite sets

Define the bijection $\mathbb{N} \to \mathcal{P}_{\text{fin}}\mathbb{N}, n \mapsto e_n$ by:

$$e_n = p \quad \text{iff} \quad n = \sum_{i \in p} 2^i.$$

Coding continuous functions II

Coding

For a Scott-continuous function $F\colon \mathcal{P}\omega \to \mathcal{P}\omega$, define

$$\mathrm{code}(F) = \{\langle m, n\rangle \mid n \in F(e_m)\}.$$

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Decoding

For $A, B \subseteq \mathbb{N}$, define:

$$A \cdot B = \{n \in \mathbb{N} \mid \exists m \in \mathbb{N}(\langle m, n\rangle \in A \text{ and } e_m \subseteq B)\}.$$

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$$A \cdot B = \{n \in \mathbb{N} \mid \exists m \in \mathbb{N}(\langle m, n\rangle \in A \text{ and } e_m \subseteq B)\}.$$

- If F is Scott-continuous, then $\mathrm{code}(F) \cdot B = F(B)$;
- The function $\mathcal{P}\omega \times \mathcal{P}\omega \to \mathcal{P}\omega, (A, B) \mapsto A \cdot B$ is itself Scott-continuous.

Combinatory completeness

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By repeatedly applying code$(-)$, we obtain $A \subseteq \mathbb{N}$ such that:

$$AB_1 \cdots B_n = t(B_1, \ldots, B_n)$$

for all $B_1, \ldots, B_n \subseteq \mathbb{N}$.

Examples of (non-)continuous functions

Example

The following functions are Scott-continuous ('computable'):

- $(A, B) \mapsto A \cup B$;
- $(A, B) \mapsto A \cap B$;
- $A \mapsto$ the closure of A under finite sums.

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Non-example

The function $A \mapsto \mathbb{N} - A$ is not Scott-continuous.

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Oracle functions

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We consider functions $F\colon \mathcal{B} \to \mathcal{B}$ where the input acts as an **oracle**, i.e., a resource that may be consulted finitely many times.

Example of an oracle function

Example

The function $F\colon \mathcal{B} \to \mathcal{B}$ is defined as follows. If $\beta \in \mathcal{B}$ and $n \in \mathbb{N}$, we define $x_0, x_1, \ldots, x_n \in \mathbb{N}$ by:

$$x_0 = n, \quad x_{i+1} \simeq x_i + \beta(x_i),$$

and we set $F(\beta)(n) \simeq x_n$.

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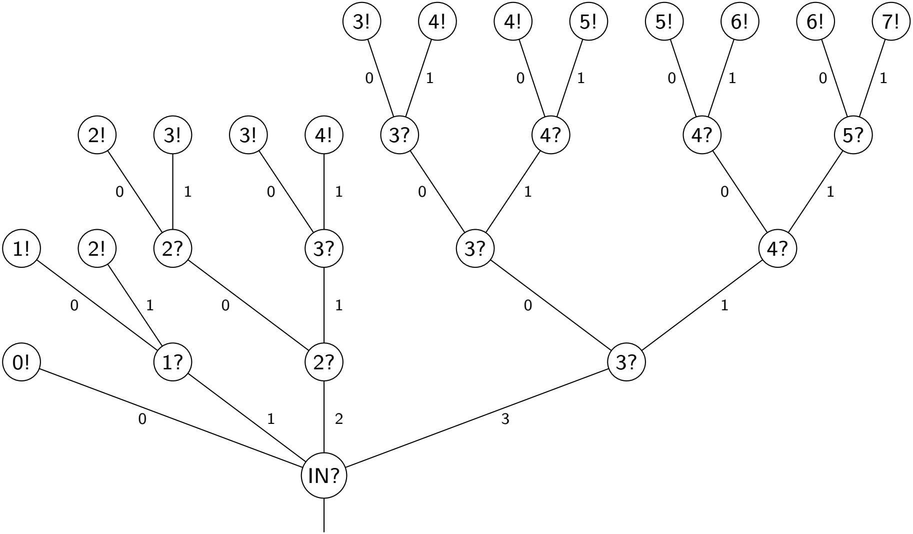
and we set $F(\beta)(n) \simeq x_n$.

- $F(\beta)(0) = 0$
- $F(\beta)(1) \simeq 1 + \beta(1)$
- $F(\beta)(2) \simeq 2 + \beta(2) + \beta(2 + \beta(2))$
- $F(\beta)(3) \simeq 3 + \beta(3) + \beta(3 + \beta(3)) + \beta(3 + \beta(3) + \beta(3 + \beta(3)))$
- ...

Tree representation



Coding oracle computation I

We can describe F by the function that maps tree positions to either queries or final outcomes.

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Coding of queries or outcomes

- we represent the query

  

  by $2q + 1$;
- we represent the outcome

  

  by $2m$.

Coding oracle computation II

Tree positions are really just finite sequences, telling us how we got there.

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Coding of finite sequences

Define the injective function $\mathbb{N}^* \to \mathbb{N}, (a_1, a_2, \ldots, a_k) \mapsto [a_1, a_2, \ldots, a_k]$ by:

$$[a_1, a_2, \ldots, a_k] = \prod_{i=1}^{k} p_i^{a_i+1} = 2^{a_1+1} 3^{a_2+1} \cdots p_k^{a_k+1},$$

where p_i is the i^{th} prime number.

Coding oracle computation III

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- there is $q_2 \in \mathbb{N}$ such that $\alpha([n, u_0, u_1]) = 2q_2 + 1$ and $\beta(q_2) = u_2$;

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- $\alpha([n, u_0, u_1, \ldots, u_{k-1}]) = 2m$.

This makes $\mathcal{B}$ into a partial combinatory algebra.

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Applicative morphisms

Applicative morphisms

Let A, B be PCAs. An *applicative* morphism $A \to B$ is a function $f \colon A \to \mathcal{P}_{\neq\emptyset} B$ for which there exists an $r \in B$ such that:

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Category of PCAs

This yields a category of PCAs, where:

- $\mathrm{id}_A(a) = \{a\}$;
- if $A \xrightarrow{f} B \xrightarrow{g} C$, then $gf(a) = \bigcup_{b \in f(a)} g(b)$.

Example I

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Task: find a Scott-continuous function $F\colon \mathcal{P}\omega \times \mathcal{P}\omega \to \mathcal{P}\omega$ such that $F(\mathrm{graph}(\alpha), \mathrm{graph}(\beta)) = \mathrm{graph}(\alpha \cdot \beta)$.

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Idea: the fact that $(\alpha \cdot \beta)(n) = m$ depends on only finitely many values of α and β.

Example II

Example

There is an applicative morphism $g\colon \mathcal{P}\omega \to \mathcal{B}$ given by:

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Idea: systematically inspect every element of $\mathrm{im}(\alpha)$ and every finite subset of $\mathrm{im}(\beta)$.

Comparing morphisms

Example (silly)

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Ordering on applicative morphisms

If $f, f'\colon A \to B$ are applicative morphisms, we say that $f \leq f'$ if there exists an $s \in B$ such that: if $b \in f(a)$, then $sb \in f'(a)$.

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(This makes the category of PCAs enriched over preorders.)

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For example: $G(B) = \{n \in \mathbb{N} \mid \exists m \in \mathbb{N}(\langle m, n \rangle \in B)\}$.

Thank you!

Defense: Monday 30 May at 12:15