Tame Geometry and a Finiteness Theorem for Variations of Hodge structures

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Utrecht University

Based on:

2112.06995 with Ben Bakker, Christian Schnell, Jacob Tsimerman

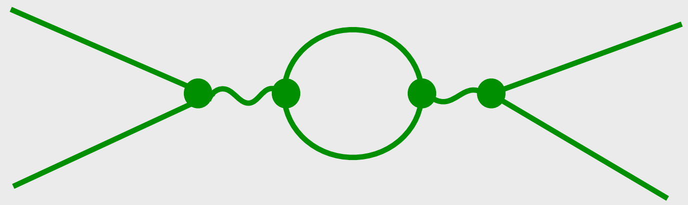
2112.08383 - Tameness Conjecture

Introduction and motivation

Corner stones of fundamental physics

- Particle physics - Standard Model of Particle Physics

→ special quantum field theory



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  (quarks, leptons, photon, gluons,…)

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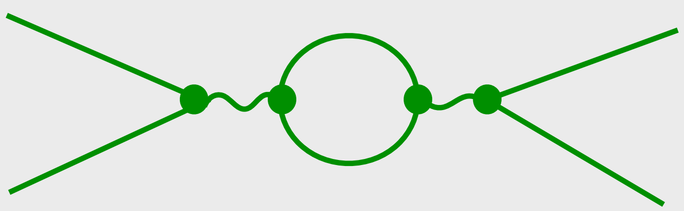

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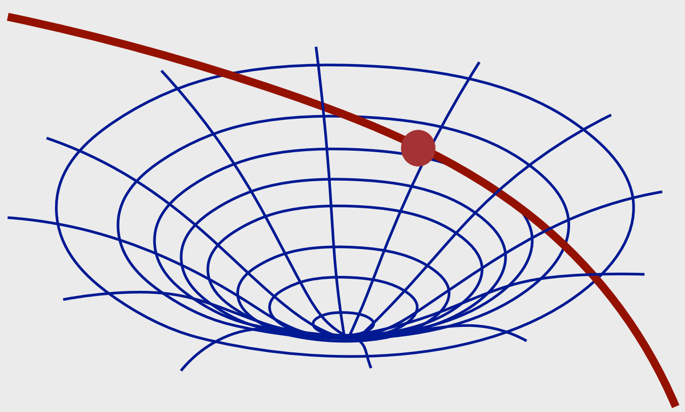
- Cosmology and Gravity - Einstein’s theory of General Relativity

→ classical theory using Riemannian geometry

- curved spaces: manifolds
- differential geometry



Quantum Gravity and its imprints

We are looking for fundamental theory unifying particle physics and General Relativity. $\rightarrow$ Theory of Quantum Gravity

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→ String Theory is a promising candidate for such a theory

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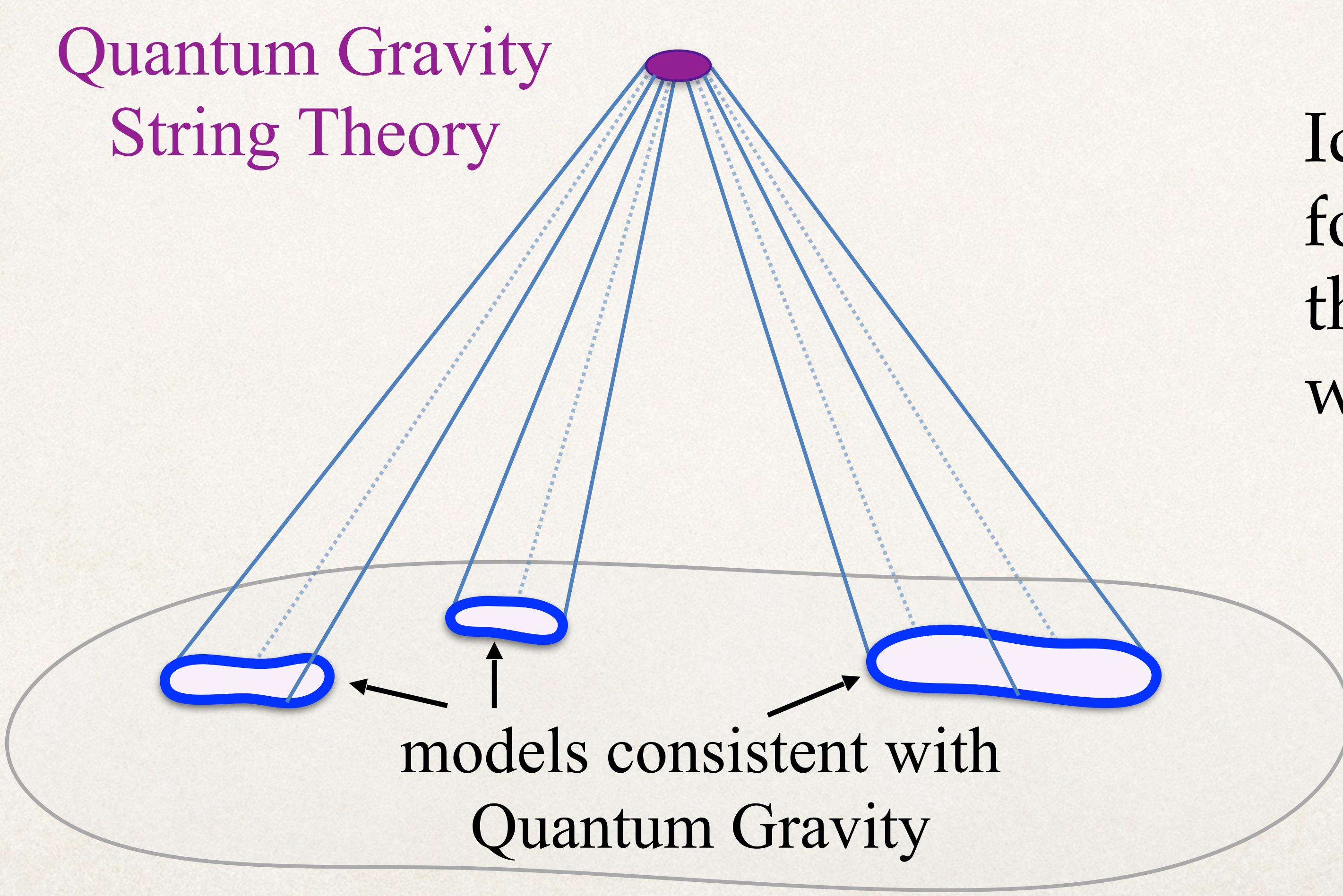
A very active research field: Swampland Program



Identify properties of four-dimensional models that make them compatible with quantum gravity.

String theory and higher dimensions

Why does string theory yield many four-dimensional models?
What has this to do with geometry?

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$\rightarrow$ Four-dimensional physics depends on choice of Y

Solutions with background fields

Problem: deformations of Y can correspond to massless fields
$\rightarrow$ fifths force $\rightarrow$ immediate contradiction with experiment

Solutions with background fields

Solution: Flux Compactifications review: [Graña] [Kachru,Douglas]

…[Becker,Becker ’96],[Gukov,Vafa,Witten ’99],[Giddings,Kachru, Polchiski ’03],[TG,Louis ’04]…

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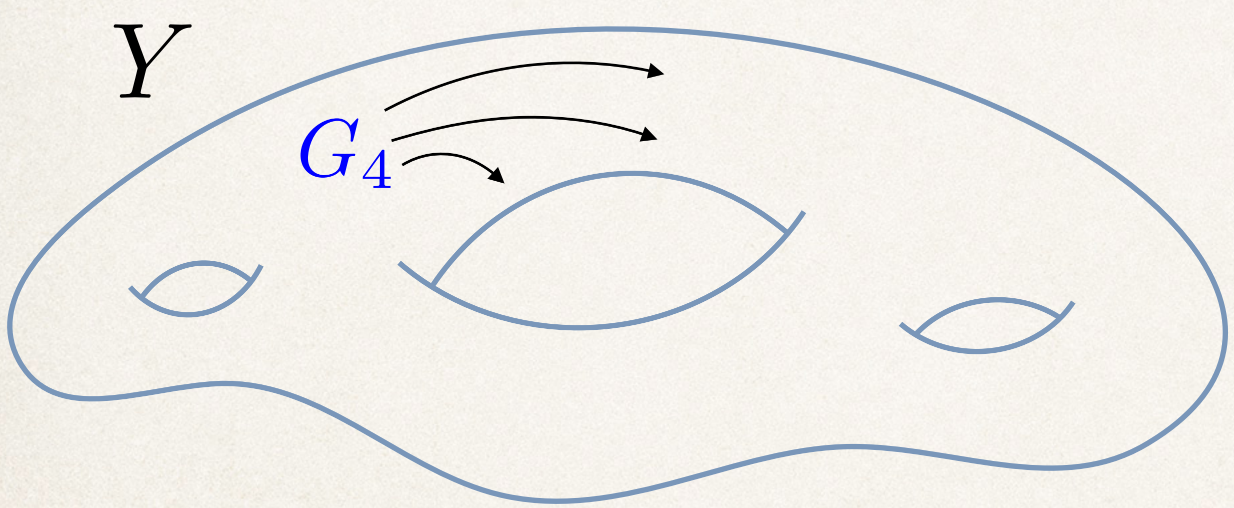
rough idea: introduce generalization of electromagnetic field, called G_4 on eight-dimensional manifold Y



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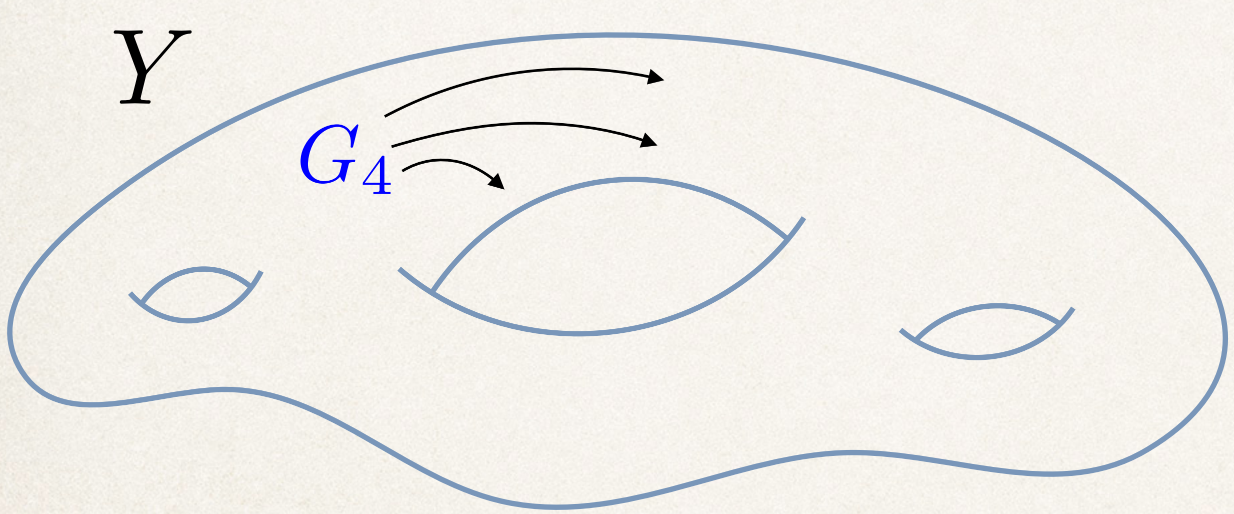

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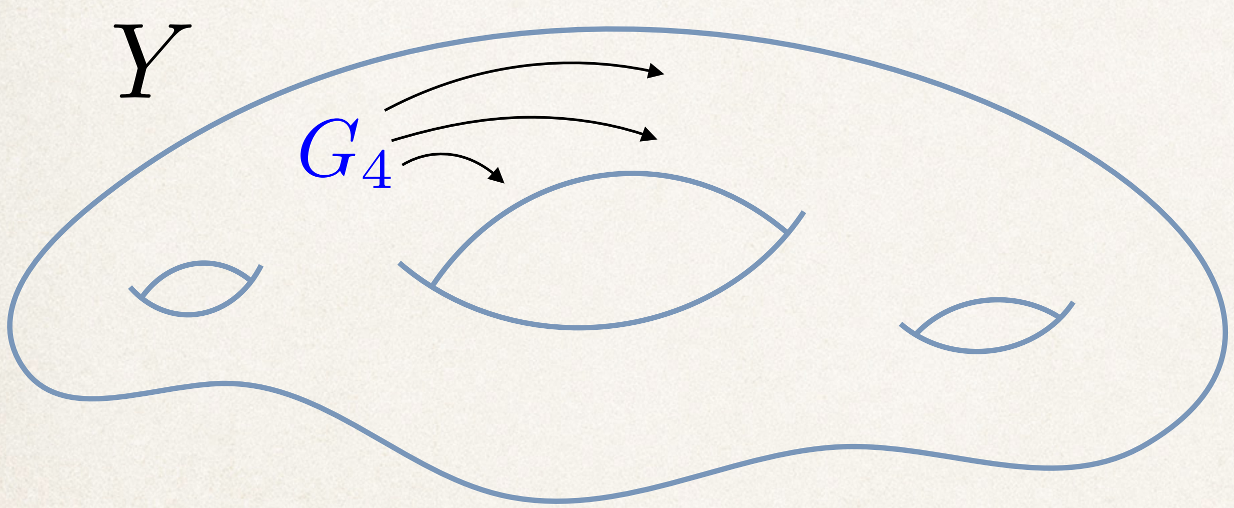

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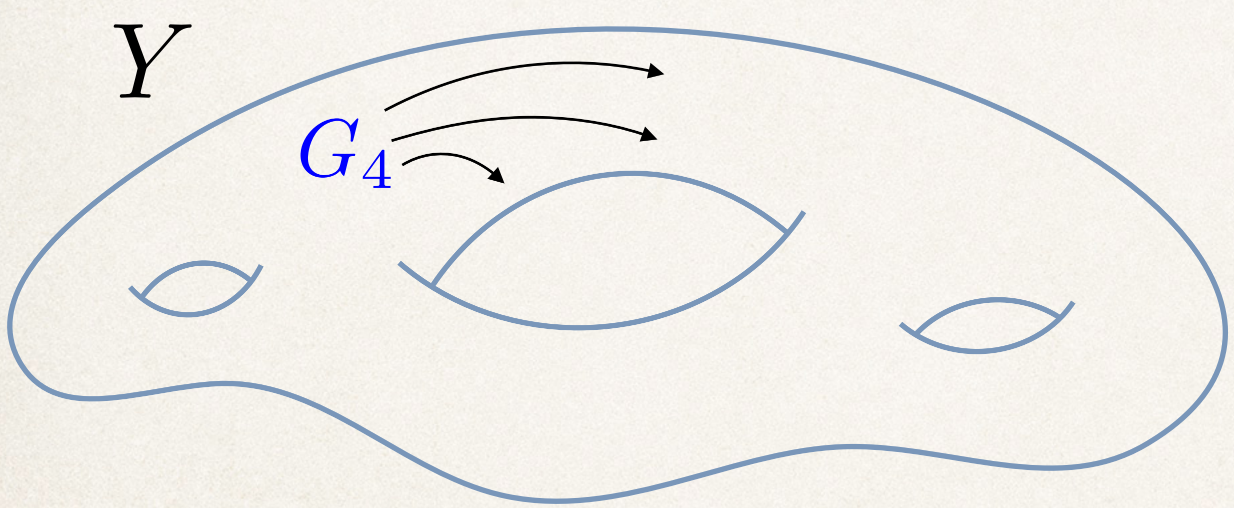

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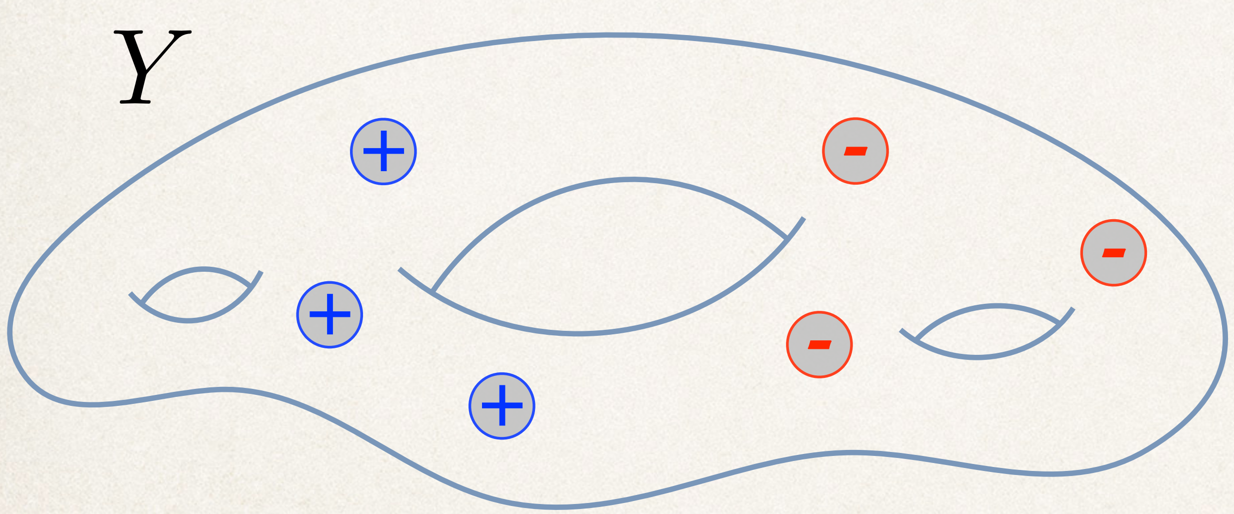

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$$n_+ + n_- = 0$$

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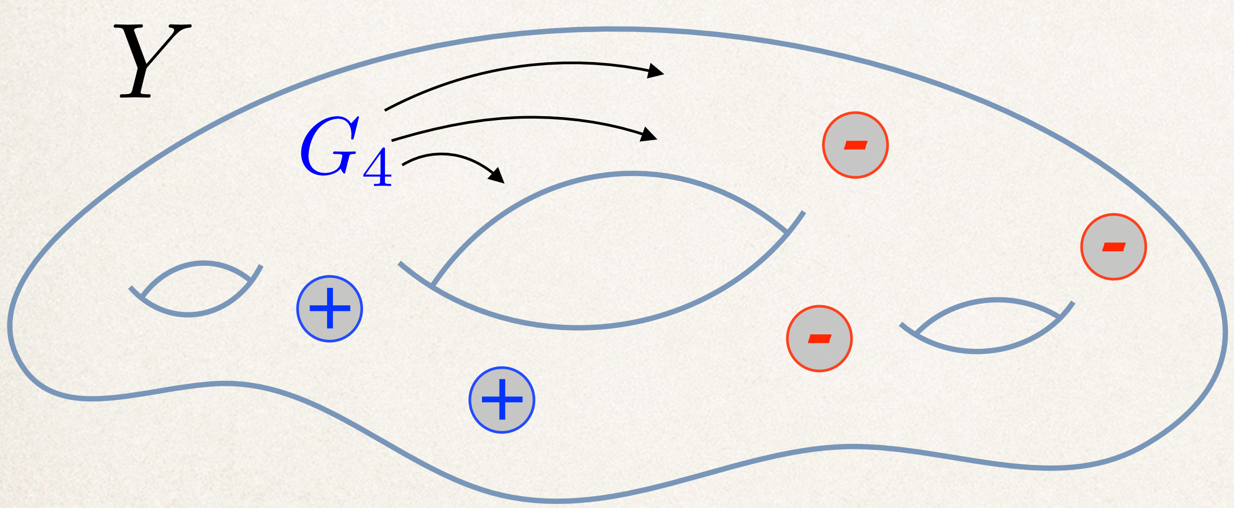

Assume Y is compact:

$$\int_Y G_4 \wedge G_4 + n_+ + n_- = 0$$

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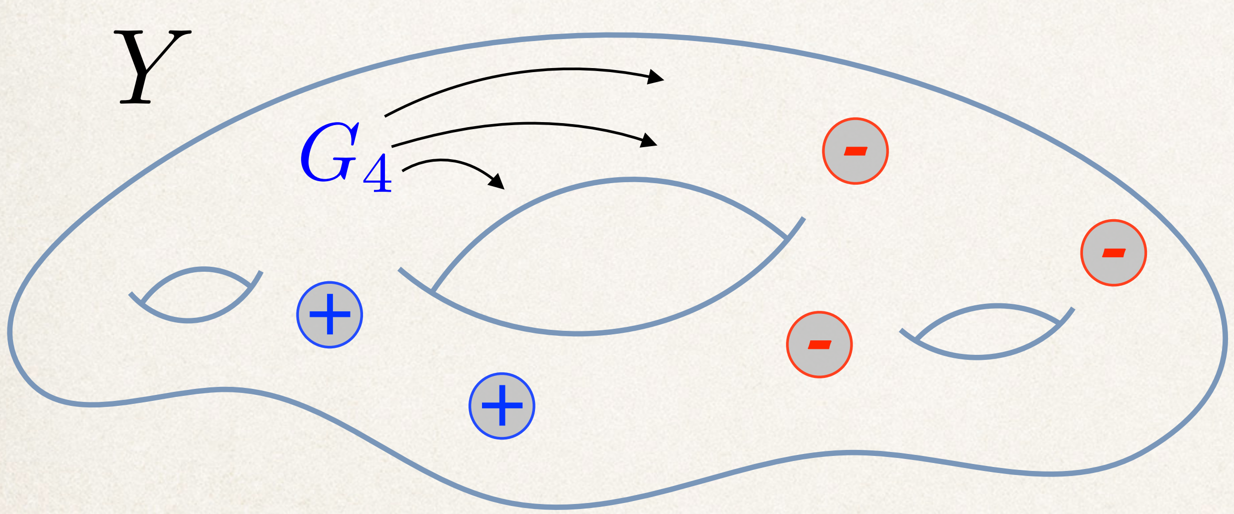

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(in cohom.) 'self-dual flux'

Hodge star operator on Y Kähler form on Y

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$\Rightarrow$ should be read as a condition on the choice of complex structure and Kähler structure $\Rightarrow$ fix deformations

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- Much more general: Is the number of four-dimensional models arising from string theory finite?

  much activity: [Vafa][Adams,DeWolfe,Taylor] [Kim,Shiu,Vafa] [Kim,Tarazi,Vafa] [Cvetic,Dierigl,Lin,Zang] [Dierigl,Heckman] [Font,Fraiman,Grana,Nunez,DeFreitas] [Hamada,Vafa] [Taylor etal] [Kim,Shiu,Vafa],[Lee,Weigand],[Tarazi,Vafa] [Hamada,Montero,Vafa,Valenzuela]

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Finiteness criterion seems to be a yes/no-criterion:
$\Rightarrow$ turn finiteness into a structural criterion: tameness conjecture

[TG '21]

Mathematical Formulation of the Problem

Calabi-Yau manifolds and moduli spaces

- Calabi-Yau manifolds of complex dimension D:

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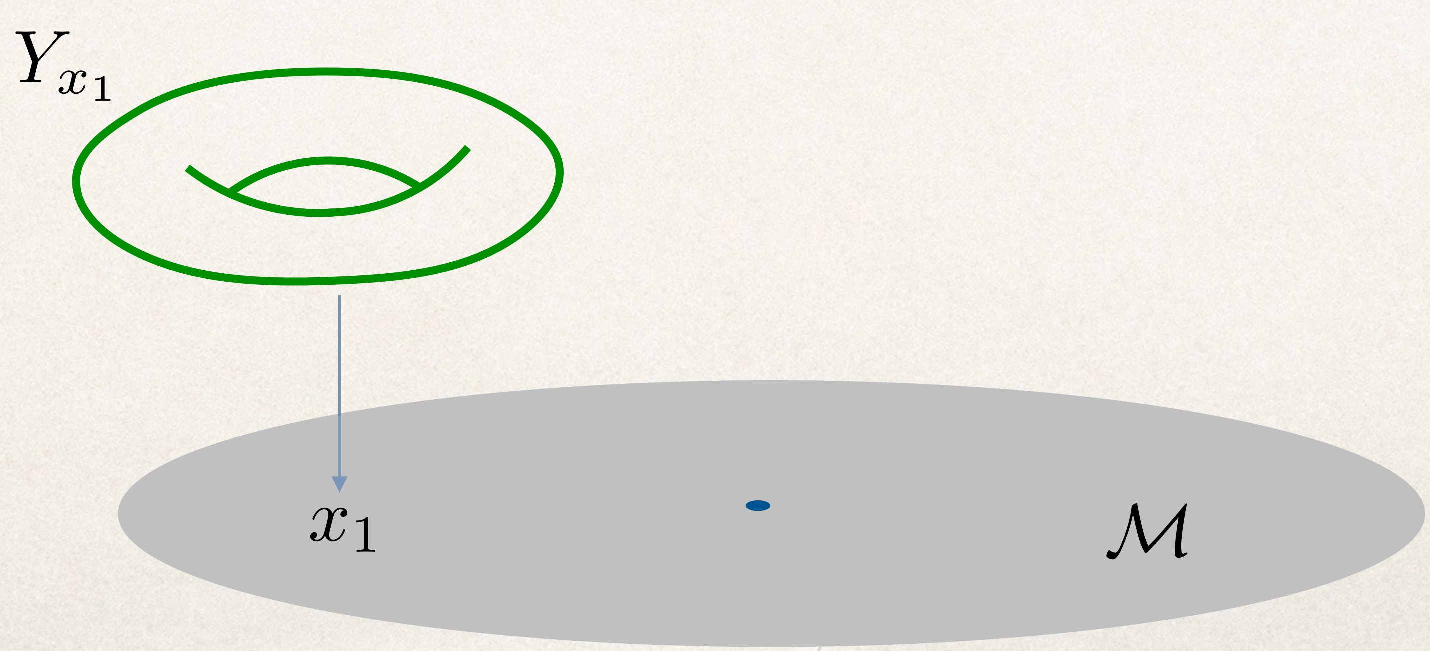

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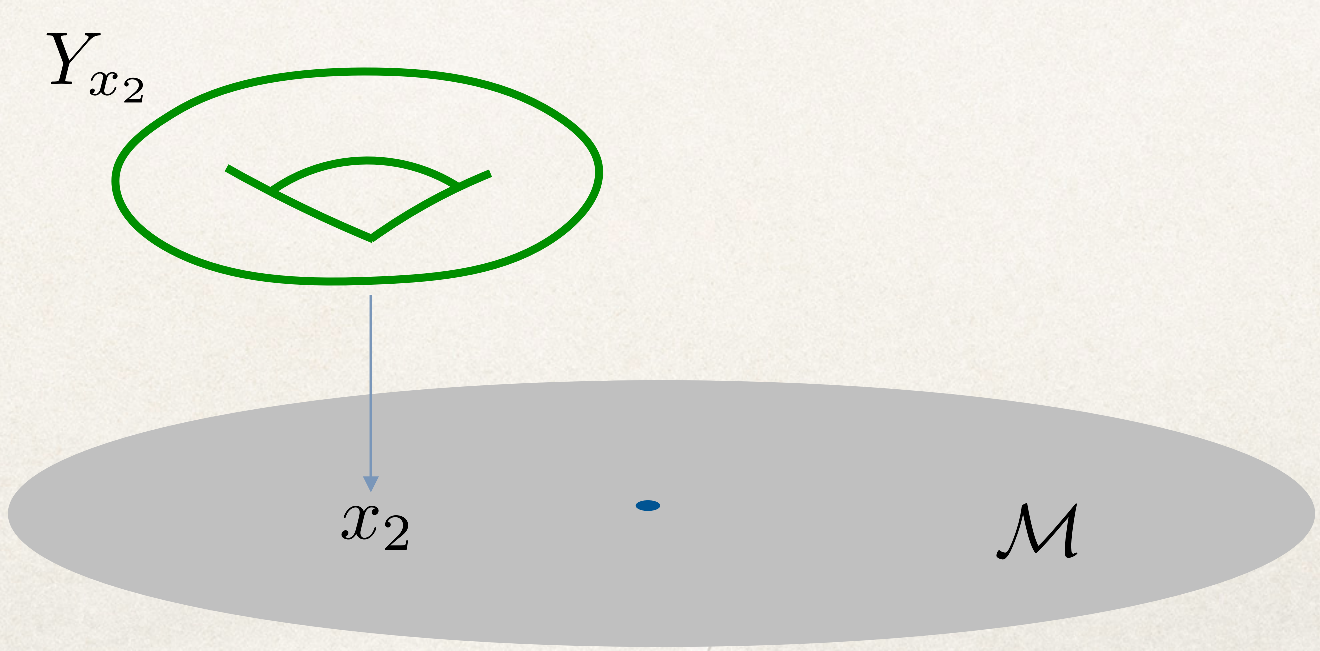

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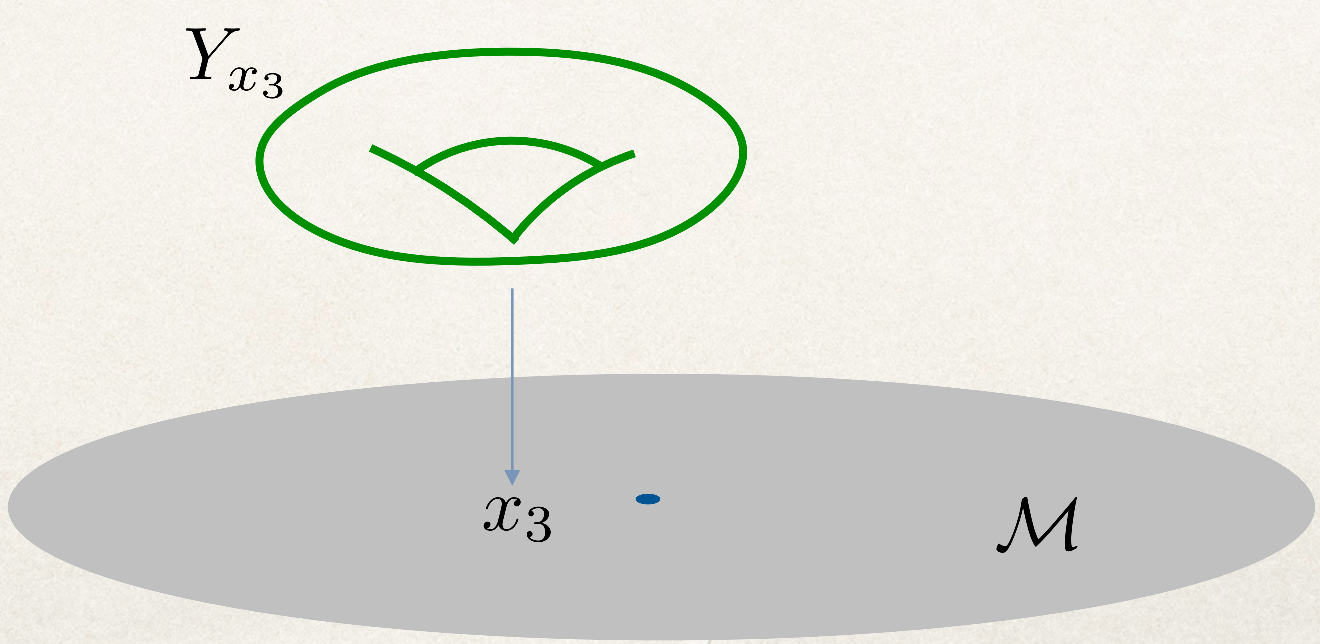

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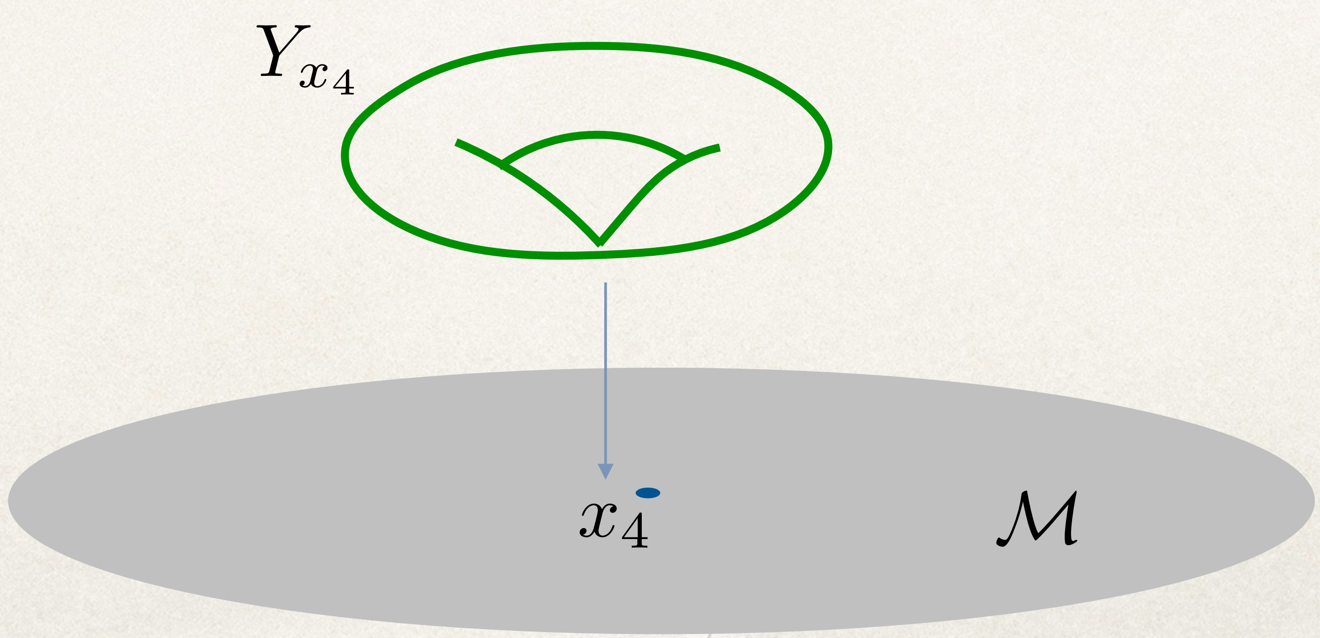

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- $\mathcal{M}$ is quasi-projective [Viehweg], can be made smooth [Hironaka]

Connection with Hodge theory

- Hodge star $*$ changes over complex structure moduli space $\mathcal{M}$

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- How to find solution? $\rightarrow$ (p,q)-forms in $H^{p,q}$ - Hodge decomposition

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$$\text{Hodge } * = \text{Weil Operator } C: \quad \begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ +1 & -1 & +1 & -1 & +1 \end{array}$$

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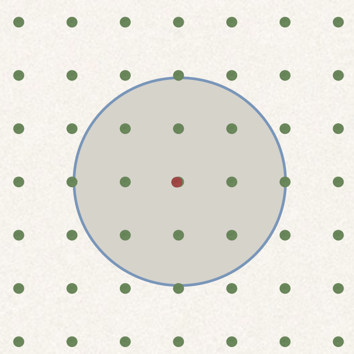
- Study how $H^{p,q}_x$ changes over the moduli space $\mathcal{M}$

  $\Rightarrow$ variations of Hodge structures

Why is finiteness non-trivial?

- Simple case: consider a fixed (p,q)-decomposition (Hodge structure)

Define: polarization $Q(v,w) := \int_Y v \wedge w$



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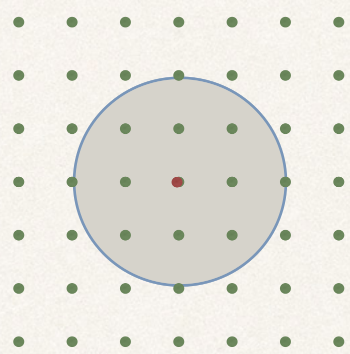

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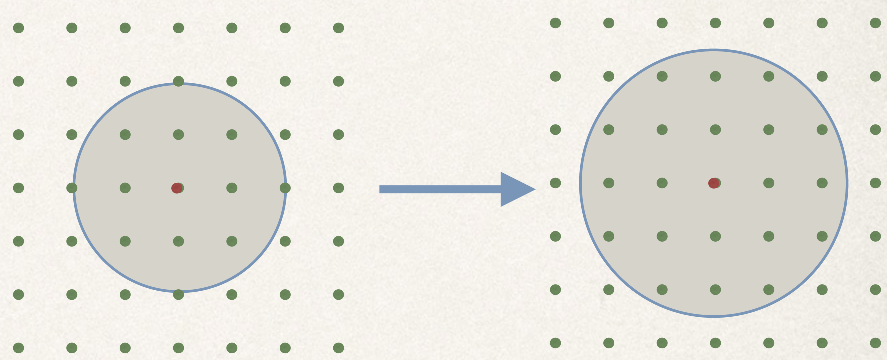

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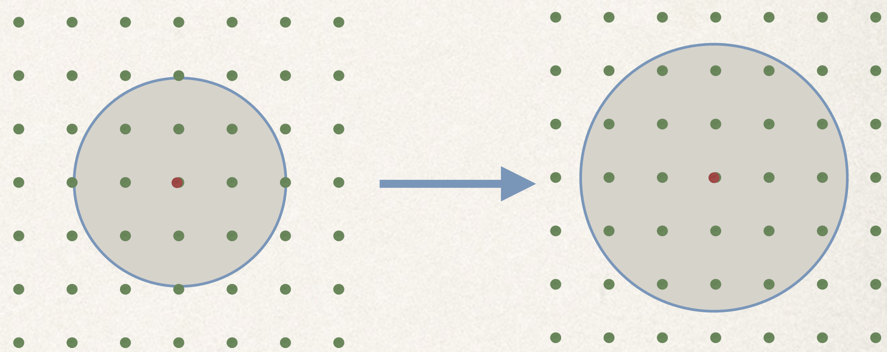

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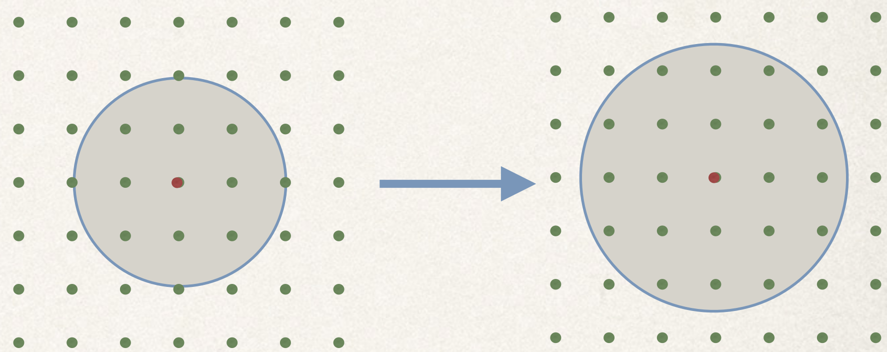

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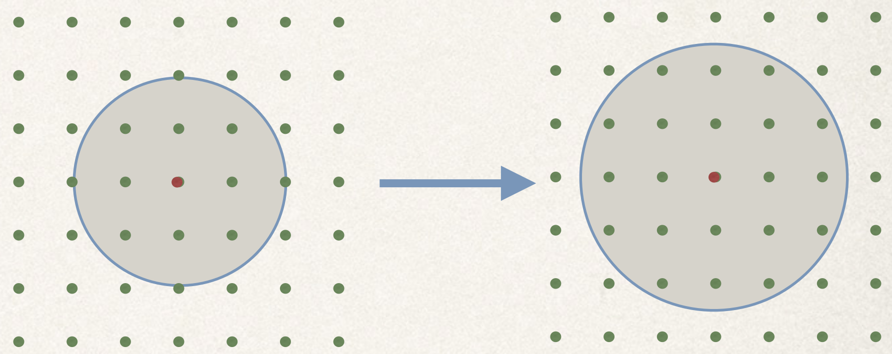

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$\Rightarrow$ works well for one-parameter limits [Schnell] [TG] '20

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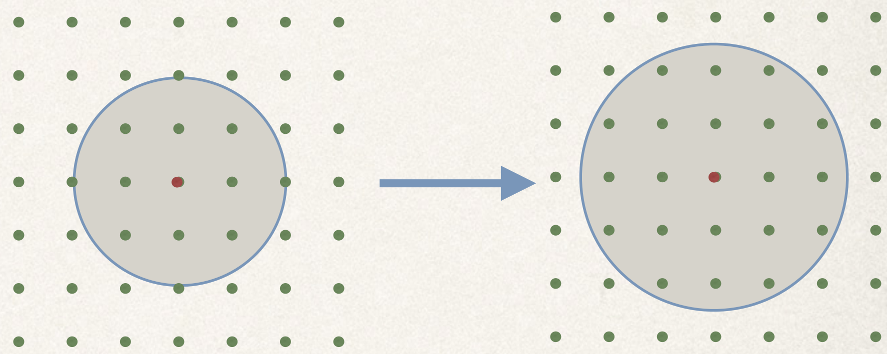

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$\Rightarrow$ using multi-variable Sl(2)-orbit theorem too involved

Theorems in Abstract Variations of Hodge Structures

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- covers the finiteness of the special case: $G_4 \; \in \; H^4(Y_4, \mathbb{Z}) \cap H^{2,2}$

(supersymmetric fluxes)

Generalization to self-dual classes

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A brief introduction to tame geometry and o-minimal structure

A mathematical structure with finiteness

- develop a mathematical framework for geometers (respect finiteness):
  - Grothendieck's dream of a tame topology [Esquisse d'un programme]
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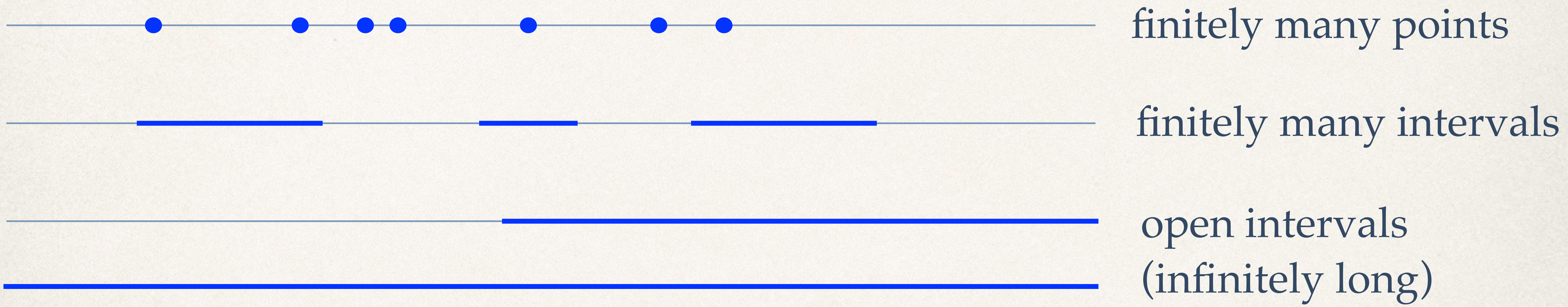
- strong finiteness properties

Finite subsets on the real line

- simplest situation: finite number of subsets of $\mathbb{R}$



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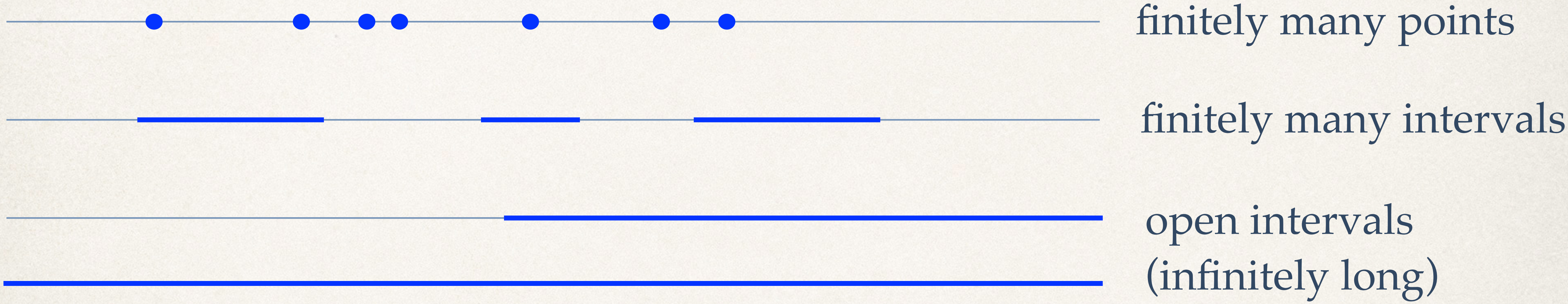

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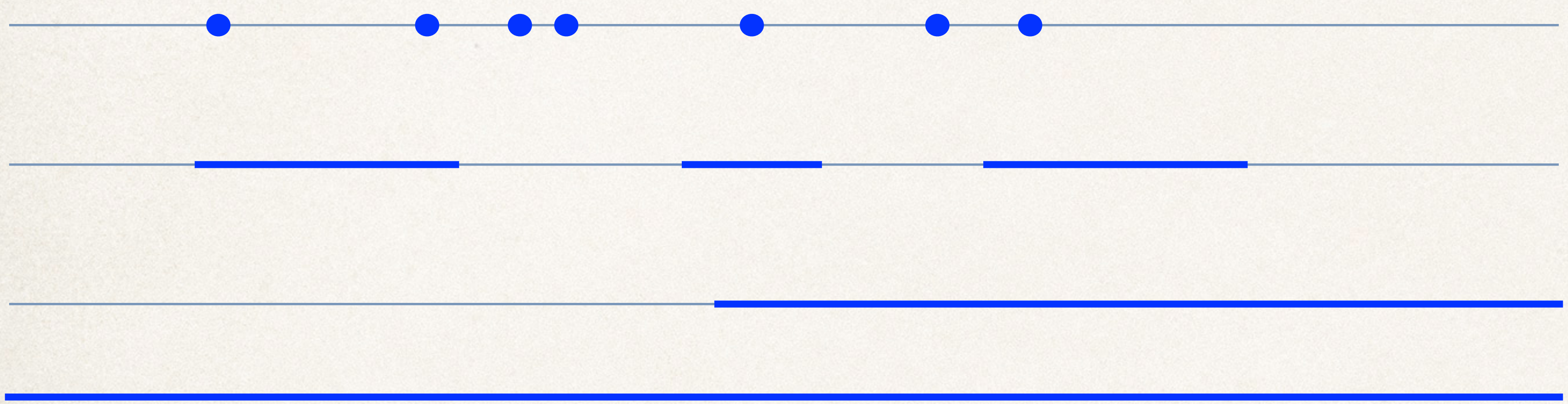

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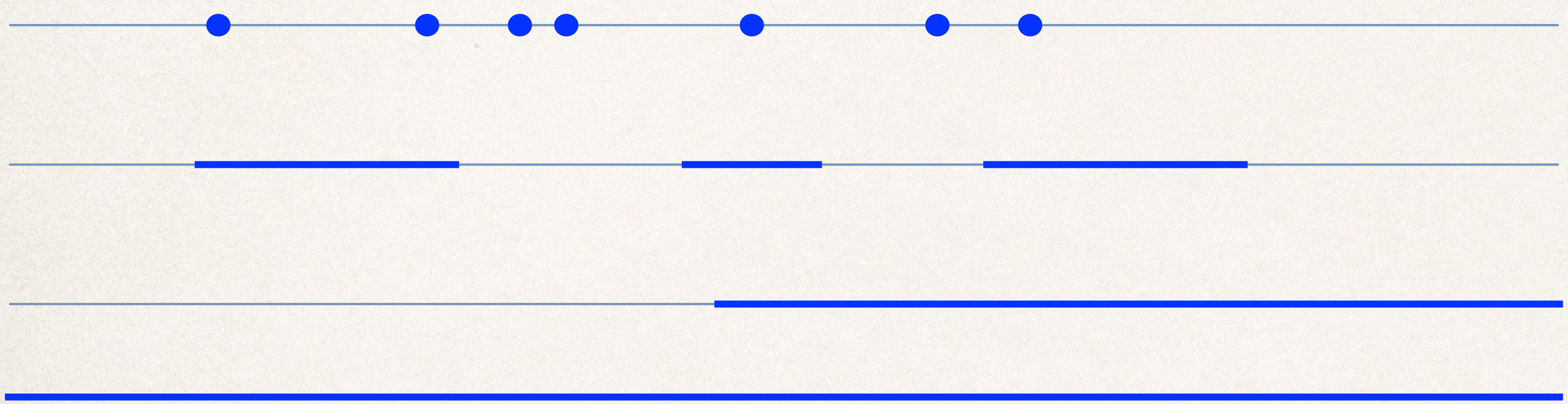

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  - projections to $\mathbb{R}$ should give the above sets
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  - extending algebraic geometry: sets defined by polynomials included
    (algebraic sets)

Tame Geometry

- Definition: An o-minimal structure $\mathcal{S}$ of sets $\{S_n\}_{n=0,1,..}$:
  - S_n are subsets of $\mathbb{R}^n$
  - S_n is closed under finite intersections, finite unions and complements
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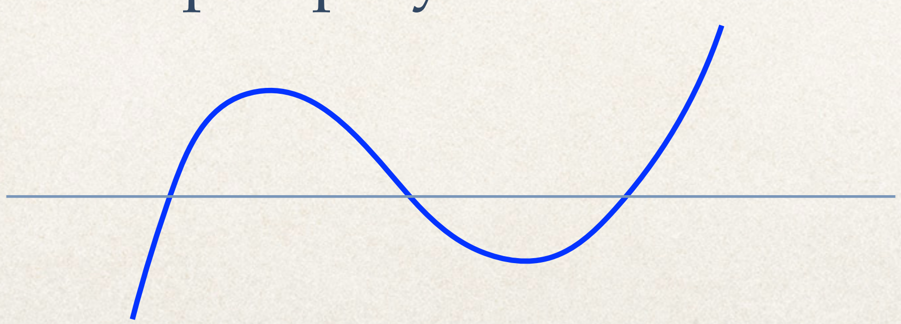
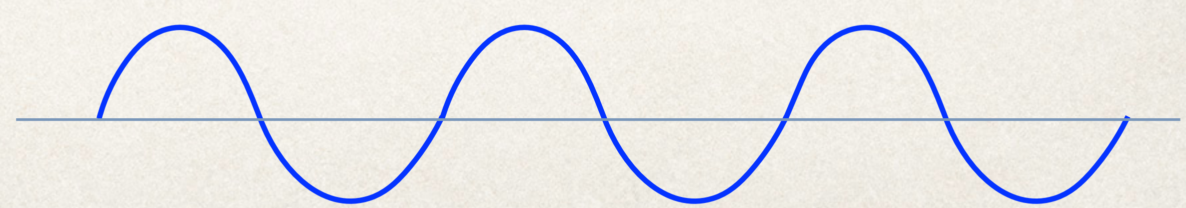
Example: polynomial function



Non-Example: $\sin(x), x \in \mathbb{R}$ is never a definable function



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- $\mathcal{S}$-definable manifold: finite definable atlas and transition functions are definable

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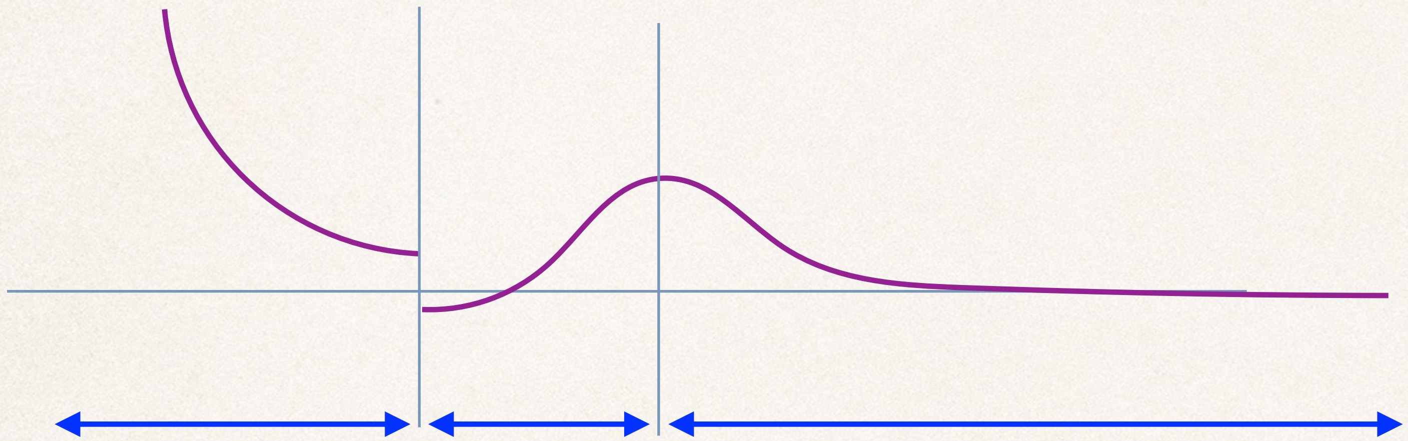

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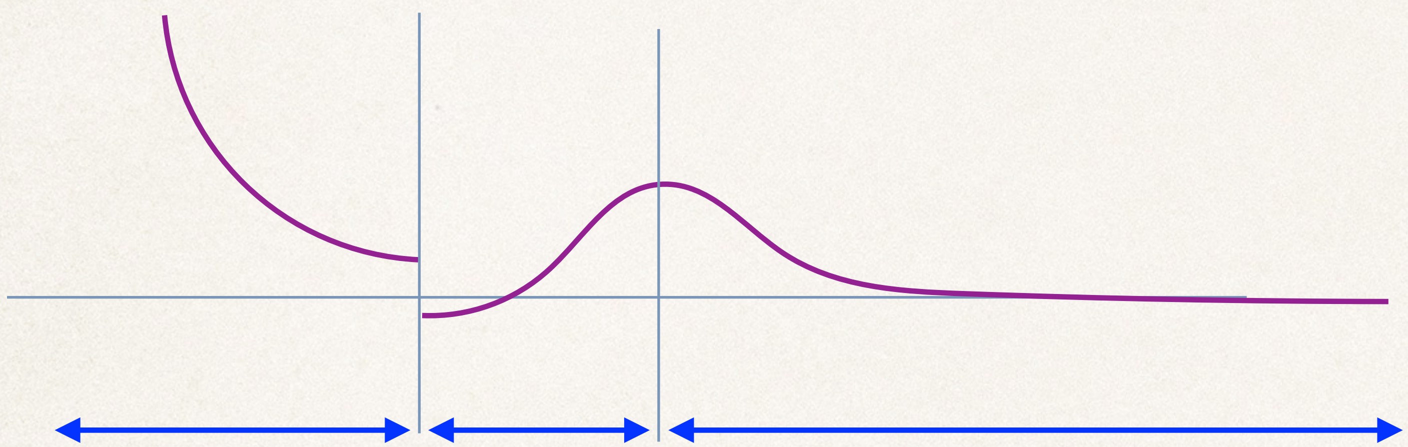

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$\to$ differentiable apart from finitely many points
$\to$ finitely many suprema and infima
$\to$ tame tail to infinity

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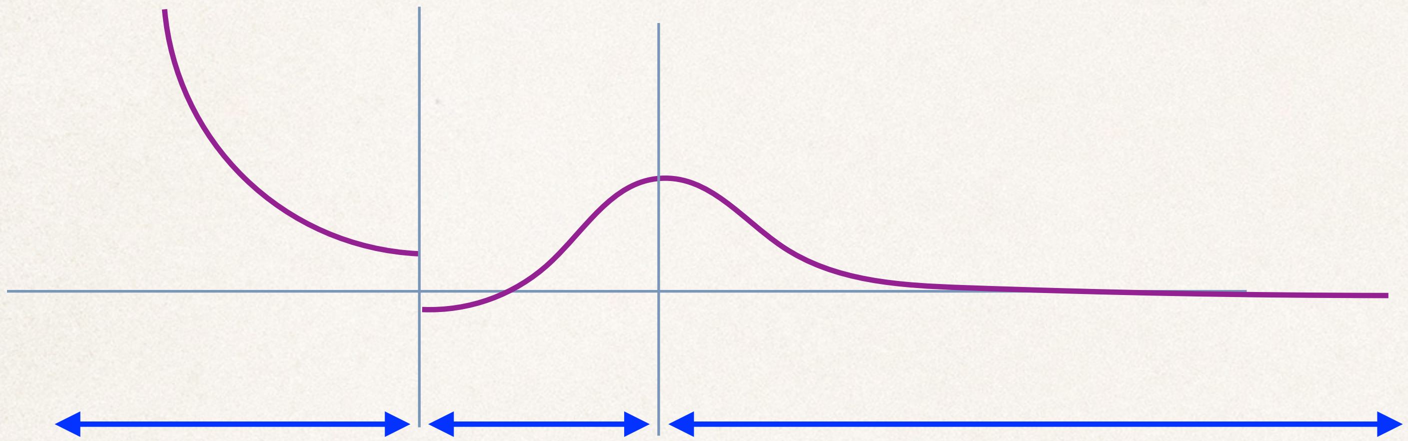

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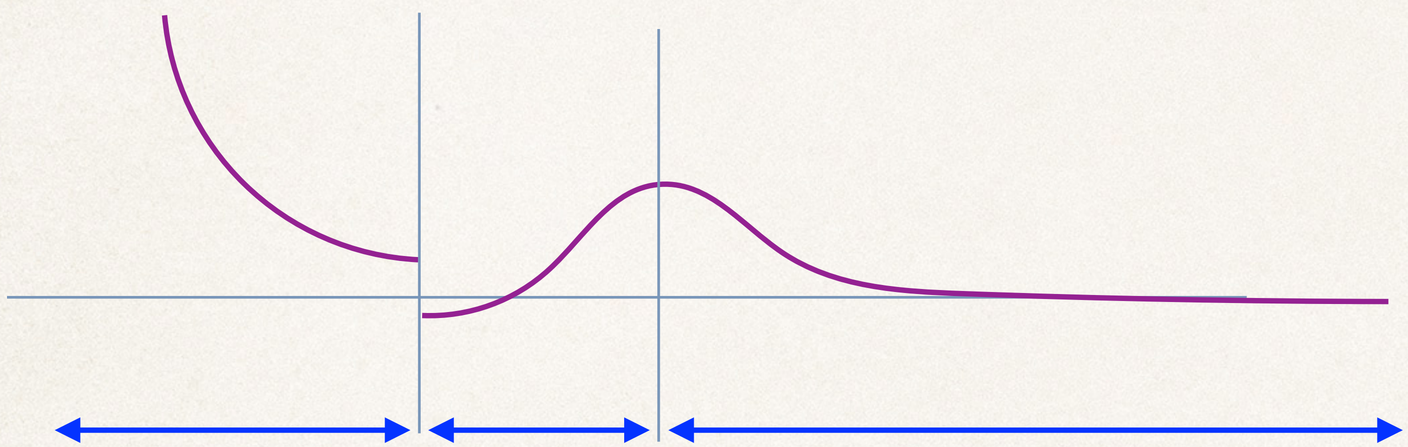

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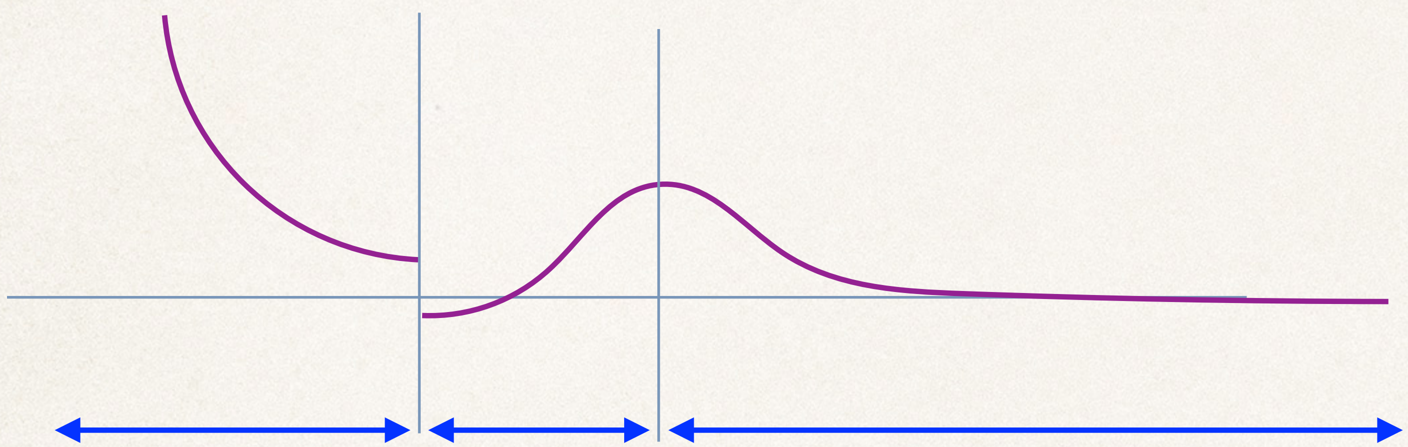

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$$e^z = e^r(\cos(\phi) + i\sin(\phi))$$

infinitely many zeros on $\mathbb{R}$

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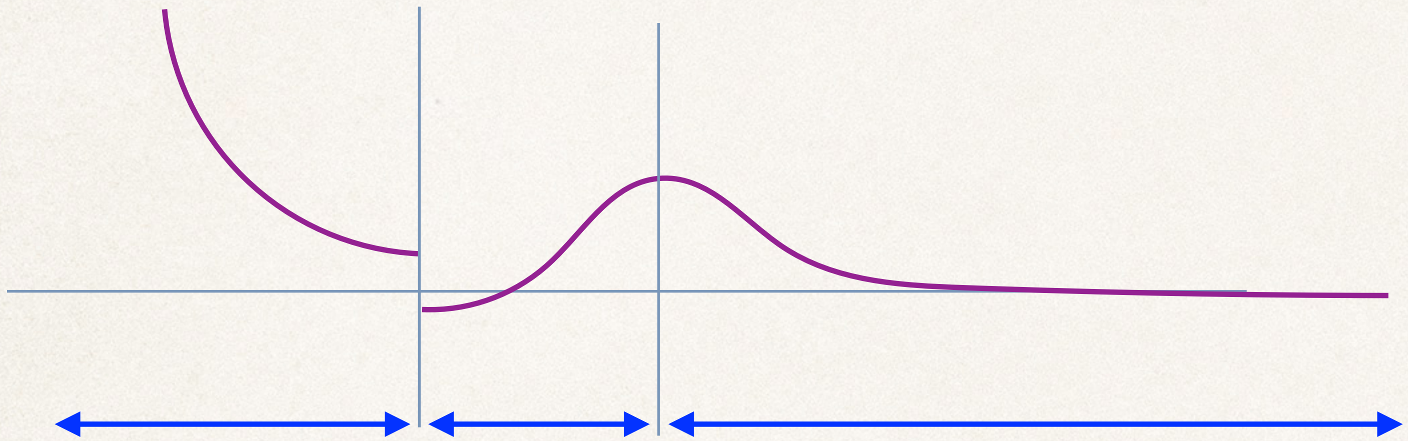

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→ restrict domain in ϕ, but e^r: does such an o-minimal structure exist

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  - examples are obtained by stating which functions are allowed to generate some of the sets $\rightarrow$ non-trivial

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    [Wilkie '96]

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  - $\mathbb{R}_{\text{alg}}$ plus exponential function: $\mathbb{R}_{\text{exp}}$ $\quad P(x_1, ..., x_n, e^{x_1}, ..., e^{x_n}) = 0$ [Wilkie '96]
  - combination of $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\text{exp}}$: $\boxed{\mathbb{R}_{\text{an,exp}}}$ [van den Dries, Miller '94]

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- Functions not definable in $\mathbb{R}_{\text{an,exp}}$ [van den Dries,Macintyre,Marker '97]
  - Gamma function: $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ restricted to $(0, \infty)$
  - Error function: $\int_0^x e^{-t^2} dt$
  - Zeta function: $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ restricted to $(1, \infty)$

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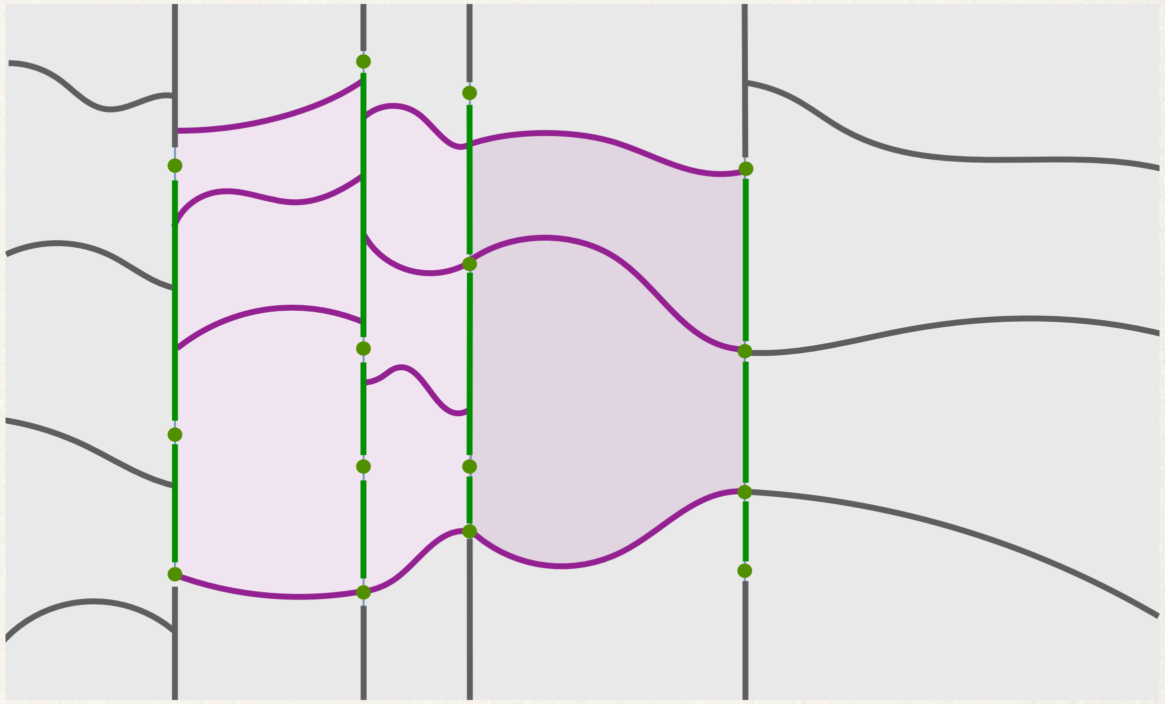

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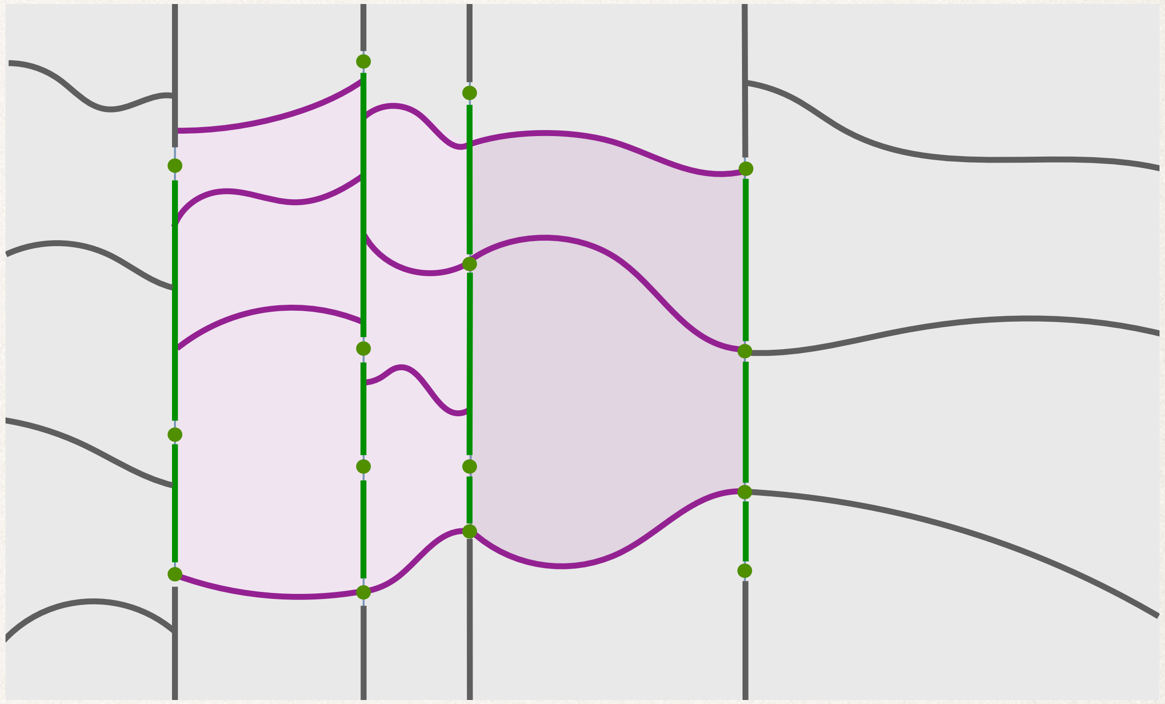

- Definability can replace compactness in many famous theorems: e.g.
  - definable Chow [Peterzil,Starchenko '06]
  - Pila-Wilkie theorem '04 (counting rational points in a definable set)

Some remarks on the proof of the finiteness theorem

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Theorem [BKT]: The Weil operator period map Φ is definable in $\mathbb{R}_{\mathrm{an,exp}}$.

$\Phi : X \to \Gamma \backslash G / K$

Γ orthogonal group of $Q(\cdot,\cdot)|_{H_{\mathbb{Z}}}$
(bigger than monodromy group)

Step 2: Extension to Hodge bundle

- Extend definability result to the Hodge bundle

Proposition: The morphism $\Phi_E : E \to \Gamma\backslash\big(G/K \times H_{\mathbb{C}}\big)$

between complex vector bundles is definable in $\mathbb{R}_{\mathrm{an,exp}}$

Proof: uses partly [Bakker,Mullane '22].

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- Idea: show that locus of self-dual classes in $\Gamma\backslash\big(G/K \times H_{\mathbb{C}}\big)$ is $\mathbb{R}_{\mathrm{alg}}$-definable using lattice theory and definablity of maps between arithmetic quotients $\to$ infer definability result for E

Step 3: Lattice reduction + single orbit

- Reduction of lattice $H_{\mathbb{Z}}$ into finitely many orbits

Theorem [e.g. Kneser]: The group Γ acts on set $\{v \in H_{\mathbb{Z}} : Q(v,v) = \ell\}$ with finitely many orbits.

- string theory consistency conditions (linked to having gravity) leads to a finiteness reduction

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Proof: some computations and definablity of $\Gamma_a\backslash G_a/K_a \to \Gamma\backslash G/K$ [BKT]

groups fixing a

A new conjecture

A new swampland conjecture

Tameness conjecture:

All effective theories valid below a fixed finite energy cut-off scale that can be consistently coupled to quantum gravity are labelled by a definable parameter space and must have scalar field spaces and coupling functions that are definable in an o-minimal structure.

Refined version:

The relevant o-minimal structure is $\mathbb{R}_{\mathrm{an,exp}}$.

A new swampland conjecture

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→ Finiteness as a structural principle in physics:

" All consistent physical theories are tame "

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- tame geometry and o-minimal structures appear to be useful in many string theory settings → tameness in physics → tameness conjecture

- How do other quantum gravity conjectures connect to tameness?
  [TG,Lanza,Li] in progress

- Evidence for tameness in quantum field theory?
  with Schlechter,… in progress

Thanks for listening!