Symmetry, Cartan Connections, and Rigidity

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1. Introduction and Motivation

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  $\rightsquigarrow$ automorphisms are determined by their finite jets in a single point
  $\rightsquigarrow$ generically there are no automorphisms
  $\rightsquigarrow$ structures with large automorphism groups or special types of automorphisms are typically geometrically and topologically constrained and hence can often be classified

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Theorem

Suppose (M^n, g) is simply-connected and $\dim(\mathrm{Iso}(M, g)) = \frac{n(n+1)}{2}$. Then (M^n, g) is isometric to either of the following spaces:

- Euclidean space $\mathbb{R}^n \cong \mathrm{Euc}(n)/O(n)$
- n-dimensional sphere $S^n \cong O(n+1)/O(n)$
- n-dimensional hyperbolic space $H^n \cong O_+(n,1)/O(n)$.

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  $\rightsquigarrow$ ϕ is determined by its 1-jet at a point.
- **Prolongation of the Killing equation**:
  Suppose $\xi = \xi^a \in \Gamma(TM)$ is a Killing vector field:
  $$\mathcal{L}_\xi g = 0 \iff \nabla_{(a}\xi_{b)} = 0,$$
  where ∇ is the Levi-Civita connection of g.

  We write $\mathfrak{inf}(M, g)$ for the Lie algebra of Killing fields and $\mathfrak{isom}(M, g)$ for the Lie algebra of Isom(M, g).

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- Solutions of $\nabla_{(a}\xi_{b)} = 0$ are in bijective correspondence to sections (ξ, μ) of $T^*M \oplus \Lambda^2 T^*M$ that are parallel with respect to the connection

$$\nabla_a^{\mathrm{prol}} \begin{pmatrix} \xi_b \\ \mu_{bc} \end{pmatrix} := \begin{pmatrix} \nabla_a \xi_b - \mu_{bc} \\ \nabla_a \mu_{bc} - R_{bc}{}^d{}_a \xi_d \end{pmatrix}.$$

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- Note that the standard fiber of the bundle $T^*M \oplus \Lambda^2 T^*M$ equals

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- $\implies$ a Killing vector field is determined by its 1-jet at a point and $\dim(\mathfrak{isom}(M,g)) \leq \dim(\mathfrak{inf}(M,g)) \leq \frac{n(n+1)}{2}$.

Suppose $(M^n, [g])$ is a conformal manifold $(n \geq 3)$
$(g \sim \hat{g} :\iff \exists f \in C^\infty_+(M, \mathbb{R})$ s.t $\hat{g} = fg)$.

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Theorem (Kobayashi 1954)

- $\mathrm{Aut}(M, [g])$ is a Lie group with

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Remark On $(\mathbb{R}^n, [g_{\mathrm{euc}}])$ all local conformal transformations are generated by translations, rotations, dilations and inversions.

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- A transformation $\phi \in \mathrm{Aut}(M,[g])$ is essential, if ϕ is not an isometry for any metric in the conformal class $[g]$.

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Lichnerowicz's Conjecture: Ferrand–Obata–Schoen Theorem

Suppose $(M^n, [g])$ is an essential conformal manifold ($n \geq 2$). Then $(M^n, [g])$ is conformally diffeomorphic to either

- $(S^n, [g_{\mathrm{rd}}])$ (if M is compact)
- $(\mathbb{R}^n, [g_{\mathrm{euc}}])$ (if M is not compact).

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A proof was given in the compact case independently by Ferrand and Obata in 70's. Later the full conjecture was proved independently by Ferrand and Schoen in the mid 90's.

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Theorem [Frances, 2012]
Let η be an infinitesimal automorphism of a conformal manifold $(M^n, [g])$ ($n \geq 3$) with a zero at $x_0 \in M$. Then:

- Either there exists an neighbourhood of x_0 on which η is inessential.
- If this is not the case, then there exists a neighbourhood of x_0 on which the geometry is locally conformally flat and η is essential.

- An infinitesimal conformal automorphism $\eta \neq 0$ has first-order zero at $x_0 \in M$, if its local flow ϕ_t fixes x_0 to first order:

$$\phi_t(x_0) = x_0 \qquad T_{x_0}\phi_t = Id : T_{x_0}M \to T_{x_0}M.$$

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Theorem [Frances–Melnick, 2013]
Suppose $(M^n, [g])$ is pseudo-Riemannian conformal manifold admitting an infinitesimal automorphism with a first-order zero $x_0 \in M$. Then there exits open subset $U \subset M$ with $x_0 \in \overline{U}$ on which $(M, [g])$ is locally conformally flat.

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Similar results are known for non-degenerate CR-structure of hypersurface type (Beloshapka/Loboda/Kruzhilin, 70's and 80's) and various parabolic geometries (Čap–Melnick, Melnick-N., Kruglikov–The,...).

2. Cartan Geometries

Klein's Erlangen Programme:

Geometric structure $\rightleftharpoons$ transitive left action $G \times M \to M$ of a Lie group G on a manifold M.

$\rightsquigarrow$ $M \cong G/P$ is a homogeneous space, where G is acting by left multiplication.

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Example: Riemannian manifolds can be seen as manifolds whose tangent space at each point has the structure of an Euclidean space $\mathrm{Euc}(n)/\mathrm{O}(n) \cong \mathbb{R}^n$, but this structure in general varies from point to point.

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- a principal G-bundle $\tilde{\mathcal{G}} \to M$ with a principal connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{g})$
- a reduction of structure group $i : \mathcal{G} \to \tilde{\mathcal{G}}$ to P

that satisfy that $\omega = i^*\tilde{\omega} \in \Omega^1(\mathcal{G}, \mathfrak{g})$ induces an isomorphism

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Homogenous model: $G \to G/P$ equipped with the Maurer–Cartan form $\omega = \omega_{MC} \in \Omega^1(G, \mathfrak{g})$.

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$\kappa \equiv 0$ if and only if the Cartan geometry is locally equivalent to its homogeneous model.

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Examples:

- Riemannian manifolds $(M^n, g) \leftrightarrow$ torsion-free Cartan geometries of type $\mathrm{Euc}(n)/O(n) \cong \mathbb{R}^n$
- Conformal manifolds $(M^n, [g]) \leftrightarrow$ normal Cartan geometries of type $\mathrm{SO}(n+1,1)/P \cong S^n$ $(n \geq 3)$
- Projective manifolds $(M^n, [\nabla]) \leftrightarrow$ normal Cartan geometries of type $\mathrm{PSL}(n+1,\mathbb{R})/P \cong \mathbb{RP}^n$
- Parabolic geometries= Cartan geometries of type (G, P), where G is a semisimple Lie group and P a parabolic subgroup.

Suppose $P \leq G$ is a parabolic subgroup of a semisimple Lie group.

- Filtered Lie algebra

$$\mathfrak{g} = \mathfrak{g}^{-k} \supset \dots \supset \mathfrak{g}^{-1} \supset \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \dots \supset \mathfrak{g}^k \quad [\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$$

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$$\mathfrak{g} \cong \mathrm{gr}(\mathfrak{g}) = \underbrace{\mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}}_{=:\mathfrak{g}_-} \oplus \mathfrak{g}_0 \oplus \underbrace{\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k}_{=:\mathfrak{p}_+} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

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- We write $G_0 = \{g \in P : \mathrm{Ad}(p)(\mathfrak{g}_i) \subset \mathfrak{g}_i \, \forall i\}$ for the group corresponding to $\mathfrak{g}_0$.

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Prolongation procdure by Tanaka, Morimto, Čap-Schichl imply: In almost all cases, a regular normal parabolic geometry is determined by its underlying regular infinitesmal flag structure.

- The curvature of a normal parabolic geometry can be viewed as P-equivariant function

$$\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \cong \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g} := \mathbb{W}.$$

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- Harmonic curvature: $\hat{\kappa} : \mathcal{G} \to \widehat{\mathbb{W}}$, where $\widehat{\mathbb{W}}$ is a completely reducible subquotient of $\mathbb{W}$. $\kappa \equiv 0 \iff \hat{\kappa} \equiv 0$.

Suppose M is connected and equipped with a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P). Consider its automorphism group

$$\mathrm{Aut}(\mathcal{G}, \omega) := \{P - \text{equiv. diffeo. } \phi : \mathcal{G} \to \mathcal{G} : \phi^* \omega = \omega\}$$

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Theorem
$\mathrm{Aut}(\mathcal{G}, \omega)$ is a Lie group of dimension $\leq \dim(G)$ with Lie algebra

$$\mathfrak{aut}(\mathcal{G}, \omega) = \{\xi \in \mathfrak{inf}(\mathcal{G}, \omega) : \xi \text{ is complete}\}.$$

The Lie bracket on $\mathfrak{aut}(\mathcal{G}, \omega)$ is mapped under j_u to:

$$|[\,,\,]| : (X, Y) \mapsto [X, Y] - \kappa(u)(X, Y) \qquad X, Y \in \mathfrak{g},$$

where $j_u : \mathfrak{inf}(\mathcal{G}, \omega) \hookrightarrow \mathfrak{g}$ is given by $\xi \mapsto \omega(\xi(u))$ for some $u \in \mathcal{G}$.

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Čap–N. 2009: for parabolic geometries this can be further exploited to get upper bounds for the second largest possible dimension of $\dim(Aut(\mathcal{G}, \omega))$ and largest possible one of non-flat geometries.

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Kruglikov–The 2014: determined the submaximal dimension of $\mathfrak{inf}(M, \omega)$ for all complex parabolic geometries.

3. Some Local Rigidity Results

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- some techniques to study this question have been developed by Čap–Melnick (generalizing tools from confomal case by Fracnces–Melnick) and Melnick–N.

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For $\eta \in \mathfrak{inf}(M)$ write $\tilde{\eta} \in \mathfrak{inf}(\mathcal{G}, \omega)$ for its lift, and ϕ_t and $\tilde{\phi}_t$ for their respective flows. Then

$$\omega(\tilde{\eta}) : \mathcal{G} \to \mathfrak{g}$$

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- The P-orbit α of $Z := \omega(\tilde{\eta}(u_0)) \in \mathfrak{p}_+$ is independent of the choice of u_0 and called the geometric type of the zero.

- Projective structures: $G_0 = \mathsf{GL}(n, \mathbb{R})$ and $\mathfrak{g}_1 = (\mathbb{R}^n)^*$ $\rightsquigarrow 1$ geometric type of first-order zero.
- (Pseudo-)conformal structures: $G_0 = \mathsf{CO}(p, q)$ and $\mathfrak{g}_1 = (\mathbb{R}^{(p,q)})^* \rightsquigarrow 3$ geometric types of first-order zeros.

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- On $(\pi : G \to G/P, \omega_{MC})$: right-invariant vector R_Z generated by $Z \in \mathfrak{p}_+$ has first-order zero at eP, whose flow e^{tZ} acts by left multiplication on G/P.

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- Exponential coordinates around eP modelled on $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_- = \mathfrak{g}_{-1}$ induced by the map $\mathfrak{g}_- \to G/P$ given by $X \mapsto \pi(e^X)$.

- $X \in C(Z) = \{X \in \mathfrak{g}_{-1} = \mathfrak{g}_{-} : [X, Z] = 0\}$ implies

$$e^{tZ} e^{sX} = e^{sX} e^{tZ} \qquad e^{tZ} \pi(e^{sX}) = \pi(e^{sX}).$$

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- $X \in T(Z) = \{X \in \mathfrak{g}_{-1} : A := [Z, X] \in \mathfrak{g}_0, (X, A, Z)\, \mathfrak{sl}_2\text{-triple}\}$ implies

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- Frances–Melnick, 2013: The Cartan connection gives (via its exponential map) rise to coordinates around x_0 modelled on $\mathfrak{g}_-$ ($X \mapsto \pi(\exp(u_0, X))$) in which the action of the flow ϕ_t of η looks similar to the action of e^{tZ} around eP.

Melnick–Čap, 2013:

- $N = \pi(\exp(u_0, C(Z))) \subset M$ submanifold through x_0 of first-order zeros of same type as x_0 .
- Family of distinguished curves

$$\mathcal{T}(\alpha) = \{\gamma_X(s) = \pi(\exp(u_0, sX)) : X \in T(Z)\}$$

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$\rightsquigarrow$ restrictions on the values of κ along distinguished curves $\mathcal{T}(\alpha)$, since κ is $\tilde{\phi}_t$-invariant and P-equivariant.

Theorem [Melnick–N., 2016]

Suppose $(\mathcal{G} \to M, \omega)$ is a normal irreducible parabolic geometry of type G/P with G simple. Let $\eta \in \mathfrak{inf}(M)$ with first-order zero at $x_0 \in M$ and geometric type α. Then:

- $\hat{\kappa}$ vanishes along all curves in $\mathcal{T}(\alpha)$.
- $\kappa(x_0) = 0$.

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Theorem [Melnick–N., 2016]
If α is the minimal (nontrivial) or the open P-orbit in $\mathfrak{p}_+$, then the geometry is flat on an open set with x_0 in its closure.

4. Global Rigidity Results

Suppose (M, g) is a Riemannian manifold and consider projective structure $[\nabla]$ on M induced by the Levi-Civita connection ∇ of g.

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$$\mathrm{Iso}(M,g) \subseteq \mathrm{Aff}(M,g) \subseteq \mathrm{Proj}(M,g),$$

and we denote by subscript 0 the connected components of the identity of these groups.

Projective Lichnerowicz Conjecture

Let (M, g) be a complete connected Riemannian manifold of dimension $n \geq 2$. Then $\mathrm{Aff}_0(M,g) = \mathrm{Proj}_0(M,g)$ unless (M^n, g) is isometric to a finite quotient of $(S^n, c g_{\mathrm{rd}})$, $c > 0$.

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This conjecture was proved by V.Matveev (2007).

Theorem [Calderbank–Eastwood–Matveev–N., 2015]
Let (M, J, g) be a complete connected Kähler manifold of dimension $2n \geq 4$. Then $\mathrm{Aff}_0(J, g) = \mathrm{CProj}_0(J, g)$ unless (M, g, J) is isometric to $(\mathbb{CP}^n, J, cg_{FS})$ for some $c \in \mathbb{R}_{>0}$.

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Theorem [Matveev–N., 2017]
Suppose (M, J, g) is a connected complete Kähler manifold of dimension $2n \geq 4$, which is not isometric to $(\mathbb{CP}^n, J, cg_{FS})$ for some $c \in \mathbb{R}_{>0}$. Then the index of the subgroup $\mathsf{Aff}(J, g)$ in the group $\mathsf{CProj}(J, g)$ is at most 2.