# A cohomological proof for the integrability of strict Lie 2-algebras

Friday Fish

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# Part I

# The van Est strategy

# Integrability of Lie algebras

$\mathfrak{g}$ - Lie algebra

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## Theorem (vanEst)

*G Lie group with Lie algebra $\mathfrak{g}$ and a representation on $V$.*
*If $G$ $k$-connected,*

$$\Phi : H^n_{Gp}(G, V) \longrightarrow H^n_{CE}(\mathfrak{g}, V)$$

*isomorphism for $n \leq k$ and injective for $n = k + 1$.*

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*isomorphism for $n \leq k$ and injective for $n = k + 1$.*

- $\text{ad}(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g}) \Longrightarrow \exists$2-connected $G$ s.t. $\text{Lie}(G) = ad(\mathfrak{g})$

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$\mathfrak{g}$ - Lie algebra

$$0 \longrightarrow \mathfrak{z}(\mathfrak{g}) \hookrightarrow \mathfrak{g} \xrightarrow{\text{ad}} \text{ad}(\mathfrak{g}) \longrightarrow 0 \quad \leftrightsquigarrow \quad \begin{array}{c} [\omega_{\text{ad}}] \in H^2_{CE}(\text{ad}(\mathfrak{g}), \mathfrak{z}(\mathfrak{g})) \\ \Big\Downarrow \text{van Est} \\ \exists ! [\int \omega_{\text{ad}}] \in H^2_{Gp}(G, \mathfrak{z}(\mathfrak{g})) \end{array}$$

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$$\leftrightsquigarrow$$

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# The necessary ingredients

- Global and infinitesimal cohomology theories (classifying extensions)
- A van Est map and theorem relating them
- A canonically associated adjoint extension
- That linear Lie algebras be integrable to 2-connected Lie groups

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**Theorem (Crainic)**

*Let* $0 \longrightarrow L \longrightarrow \mathcal{A} \longrightarrow A \longrightarrow 0$ *be an exact sequence of Lie algebroids with* $L$ *abelian.*
*If* $A$ *admits a Hausdorff integration with* 2*-connected s-fibres, then* $\mathcal{A}$ *is integrable.*

# $L_\infty$-algebras

**Theorem**

*For a vector space $\mathfrak{g}$, there is a $1-1$ correspondence*

$$\left\{\begin{array}{l}\text{Lie algebra}\\ \text{structures on } \mathfrak{g}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{dg-algebra}\\ \text{structures on } (\bigwedge^\bullet \mathfrak{g}^*, \wedge)\end{array}\right\}$$

*Moreover, this correspondence extends to an equivalence of categories.*

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*Moreover, this correspondence extends to an equivalence of categories.*

## Definition (/Proposition)

An $L_\infty$-algebra structure on a graded vector space $L = \bigoplus_{k\leq 0} L_k$ is a differential on its graded symmetric algebra $\mathrm{Sym}(L^*[-1])$.

# The Chevalley-Eilenberg complex

$$\left\{ \begin{array}{l} \text{Representations of } L \\ \text{on } V_\bullet = \bigoplus_{k\leq 0} V_k \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Differentials on} \\ \mathrm{Sym}(L^*[-1]) \otimes V_\bullet \end{array} \right\}$$

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**Theorem (Liu,Sheng,Zhang)**

$$\left\{ \begin{array}{l} \textit{1-parameter infinitesimal} \\ \textit{deformations of a} \\ \textit{Lie 2-algebra} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{2-cocycles with} \\ \textit{coefficients in the} \\ \textit{adjoint representation} \end{array} \right\}$$

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- Obvious cohomology
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**Theorem**

*Abelian extensions of Lie 2-algebras are classified by the second cohomology*

# The actual complex

$\mathfrak{g} \longrightarrow \mathfrak{h}$ - Lie 2-algebra

$W \longrightarrow V$ - 2-vector space (= abelian Lie 2-algebra)

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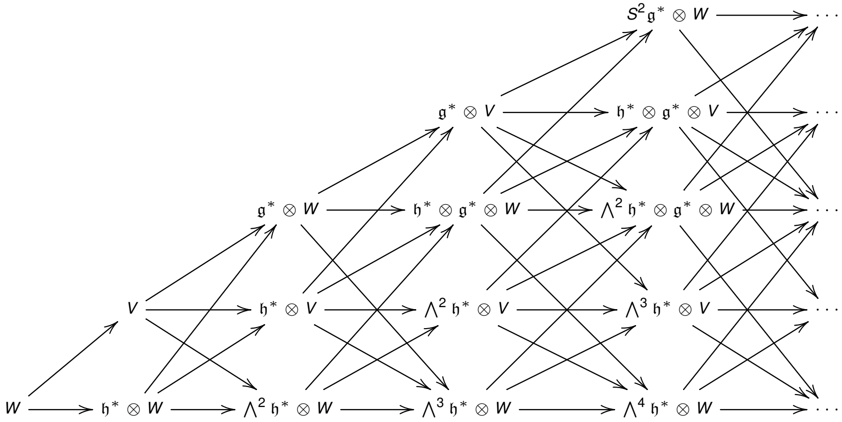

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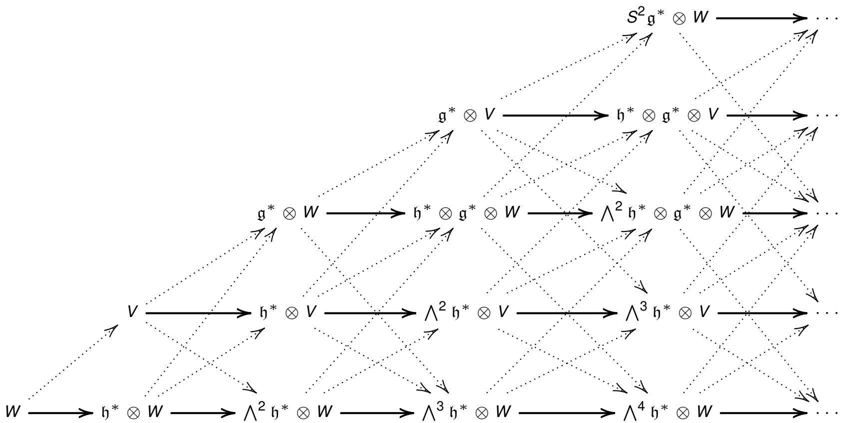

# Equivalent categories

## Definition

A crossed module of Lie algebras is a Lie algebra homomorphism $\mu : \mathfrak{g} \longrightarrow \mathfrak{h}$ together with an action by derivations $\mathcal{L} : \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ such that

$$\mu(\mathcal{L}_y x) = [y, \mu(x)]_{\mathfrak{h}} \quad \text{and} \quad \mathcal{L}_{\mu(x_0)} x_1 = [x_0, x_1]_{\mathfrak{g}}$$

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$(\mathfrak{g} \longrightarrow \mathfrak{h}, l_2, l_3)$ - $L_\infty$-algebra

$$[x_0, x_1] := l_2(\mu(x_0), x_1)$$

defines a Lie algebra structure on $\mathfrak{g}$.

# Equivalent categories

**Theorem**

*There is an equivalence of categories*

$$\left\{ \begin{array}{l} \textit{Crossed modules} \\ \textit{of Lie algebras} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{Groupoids internal} \\ \textit{to Lie algebras} \end{array} \right\}$$

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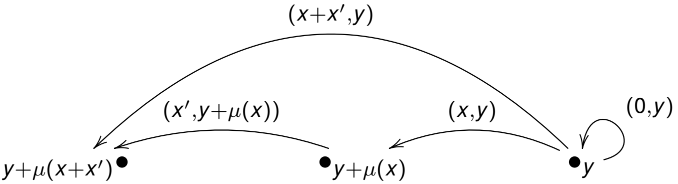

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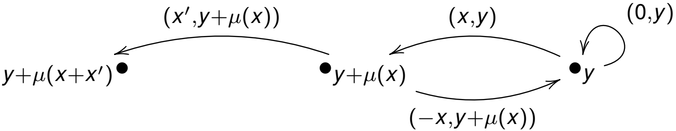

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$$t|_{\ker s} : \ker s \longrightarrow \mathfrak{h} \qquad \longleftarrow\!\!\!\!\dashv \qquad \mathfrak{g}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathfrak{h}$$

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$$\mathcal{L}_y x := [u(y), x]_{\mathfrak{g}_1} \quad \text{for} \quad y \in \mathfrak{h}, x \in \ker s$$

# The double complex and its cohomology

$\mathfrak{g}_p := \mathfrak{g}_1^{(p)} = \mathfrak{g}_1 \times_{\mathfrak{h}} \ldots \times_{\mathfrak{h}} \mathfrak{g}_1$ - the Lie algebra of $p$-composable arrows

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Consider the nerve

$$\mathfrak{h} \;\begin{matrix}\longleftarrow\\ \longleftarrow\end{matrix}\; \mathfrak{g}_1 \;\begin{matrix}\longleftarrow\\ \longleftarrow\\ \longleftarrow\end{matrix}\; \mathfrak{g}_2 \;\begin{matrix}\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow\end{matrix}\; \mathfrak{g}_3 \;\begin{matrix}\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow\end{matrix}\; \cdots$$

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Dualizing

$$C^\bullet(\mathfrak{h}) \rightrightarrows C^\bullet(\mathfrak{g}_1) \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} C^\bullet(\mathfrak{g}_2) \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}} C^\bullet(\mathfrak{g}_3) \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}} \cdots$$

# The double complex and its cohomology

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\bigwedge^3 \mathfrak{h}^* & \xrightarrow{\partial} & \bigwedge^3 \mathfrak{g}_1^* & \xrightarrow{\partial} & \bigwedge^3 \mathfrak{g}_2^* & \longrightarrow & \cdots \\
\delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \\
\mathfrak{h}^* \wedge \mathfrak{h}^* & \xrightarrow{\partial} & \mathfrak{g}_1^* \wedge \mathfrak{g}_1^* & \xrightarrow{\partial} & \mathfrak{g}_2^* \wedge \mathfrak{g}_2^* & \longrightarrow & \cdots \\
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$$

$\partial := \sum_{k=0}^{p}(-1)^k \partial_k^*$ and $\delta$ - Chevalley-Eilenberg differential

# The double complex and its cohomology

A 2-cocycle $(\omega, \varphi) \in \bigwedge^2 \mathfrak{h}^* \oplus \mathfrak{g}_1^*$

$$\begin{array}{ccccc} \delta\omega = 0 & & & & \\ \uparrow & & & & \\ \omega & \longmapsto & \partial\omega + \delta\varphi = 0 & & \\ & & \uparrow & & \\ & & \varphi & \longmapsto & \partial\varphi = 0 \end{array}$$

# The double complex and its cohomology

A 2-cocycle $(\omega, \varphi) \in \bigwedge^2 \mathfrak{h}^* \oplus \mathfrak{g}_1^*$

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{h} \oplus^{\omega} \mathbb{R} \longrightarrow \mathfrak{h} \longrightarrow 0$$

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 & & \downarrow & & \downarrow{\scriptstyle \mu_\varphi} & & \downarrow{\scriptstyle \mu} & & \\
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\end{array}$$

where $\mu_\varphi(x) := (\mu(x), -\varphi(x, 0))$

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## Theorem

*$H^2_{tot}(\bigwedge^q \mathfrak{g}_p^*)$ classifies Lie 2-algebra extensions of $\mathfrak{g}_1$ by $\mathbb{R} \rightrightarrows \mathbb{R}$.*

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Moreover,

**Theorem**

*If $\rho$ is a representation of $\mathfrak{h}$ on $V$ that vanishes on the ideal $\mu(\mathfrak{g})$, $H^2_{tot}(\bigwedge^q \mathfrak{g}_p^* \otimes V)$ classifies Lie 2-algebra extensions of $\mathfrak{g}_1$ by $V \rightrightarrows V$ .*

# Definition(s)

**Definition**

A strict Lie 2-group is a groupoid object internal to the category of Lie groups.

$\Updownarrow$

**Definition**

A crossed module of Lie groups is a Lie group homomorphism $i : G \longrightarrow H$ together with an right action by Lie group automorphisms $H \longrightarrow Aut(G)$ such that

$$i(g^h) = h^{-1} i(g) h \quad \text{and} \quad g_1^{i(g_2)} = g_2^{-1} g_1 g_2$$

for all $h \in H$ and $g, g_1, g_2 \in G$.

# The equivalence (suite)

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*There is an equivalence of categories*

$$\left\{ \begin{array}{l} \textit{Crossed modules} \\ \textit{of Lie groups} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{Groupoids internal} \\ \textit{to Lie groups} \end{array} \right\}$$

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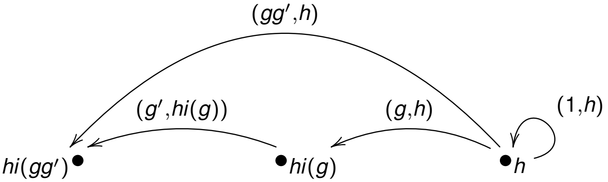

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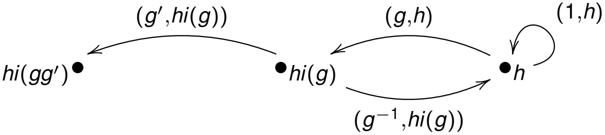

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$$g^h := u(h)^{-1} \boxtimes g \boxtimes u(h) \quad \text{for} \quad h \in H, g \in \ker s$$

# The double complex of a Lie 2-group

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
C(H^3, V) & \xrightarrow{\partial} & C(\mathcal{G}^3, V) & \xrightarrow{\partial} & C(\mathcal{G}_2^3, V) & \longrightarrow & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
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$\partial := \sum_{k=0}^{p}(-1)^k \partial_k^*$ and $\delta$ - standard group differential

# The double complex of a Lie 2-group

**Upshot**

$H^2_{tot}(C(\mathcal{G}^q_p, V))$ *classifies Lie 2-group extensions of* $\mathcal{G}$ *by* $V \rightrightarrows V$ *!*

# The van Est map

Assembling usual van Est maps

$$\Phi_p : C(\mathcal{G}_p^\bullet, V) \longrightarrow \bigwedge^\bullet \mathfrak{g}_p^* \otimes V$$

column-wise yields a map of double complexes

$$\Phi : C_{tot}(\mathcal{G}, V) \longrightarrow C_{tot}(\mathfrak{g}_1, V)$$

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$$\begin{array}{ccc} H^2_{tot}(C(\mathcal{G}_p^q, V)) & \xrightarrow{\Phi} & H^2_{tot}(\bigwedge^\bullet \mathfrak{g}_p^* \otimes V) \\ \updownarrow & & \updownarrow \\ \left\{ \begin{array}{l} \text{Extensions of } \mathcal{G} \\ \text{by } V \rightrightarrows V \end{array} \right\} & \xrightarrow{\text{Lie}} & \left\{ \begin{array}{l} \text{Extensions of } \mathfrak{g}_1 \\ \text{by } V \rightrightarrows V \end{array} \right\} \end{array}$$

# Rephrasing van Est theorem

Let $\Phi : (A^\bullet, d_A) \longrightarrow (B^\bullet, d_B)$ be a map of complexes
The mapping cone of $\Phi$ is the complex

$$C^k(\Phi) := A^{k+1} \oplus B^k \quad \text{together with} \quad d_\Phi = \begin{pmatrix} -d_A & 0 \\ \Phi & d_B \end{pmatrix}$$

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## Proposition

*The following are equivalent*

- $H^n(\Phi) = (0)$ *for* $n \leq k$
- *The induced map* $\Phi : H^n(A) \longrightarrow H^n(B)$ *is an isomorphism for* $n \leq k$ *and injective for* $n = k + 1$*.*

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- *The induced map* $\Phi : H^n(A) \longrightarrow H^n(B)$ *is an isomorphism for* $n \leq k$ *and injective for* $n = k + 1$.

## Proof.

$0 \longrightarrow B^\bullet \longrightarrow C^\bullet(\Phi) \longrightarrow A^\bullet[1] \longrightarrow 0$ □

# Rephrasing van Est theorem

**Proposition**

*The following are equivalent*

- $H^n(\Phi) = (0)$ *for* $n \leq k$
- *The induced map* $\Phi : H^n(A) \longrightarrow H^n(B)$ *is an isomorphism for* $n \leq k$ *and injective for* $n = k+1$*.*

**Proof.**

$0 \longrightarrow B^\bullet \longrightarrow C^\bullet(\Phi) \longrightarrow A^\bullet[1] \longrightarrow 0$ inducing

$$\cdots \to H^k(B) \longrightarrow H^k(\Phi) \longrightarrow H^k(A^\bullet[1]) \xrightarrow{\Phi^*} H^{k+1}(B) \to \cdots$$

□

# A van Est theorem

$\Phi : A^{\bullet,\bullet} \longrightarrow B^{\bullet,\bullet}$ is a map of double complexes if and only if $C^{\bullet}(\Phi_0) \longrightarrow C^{\bullet}(\Phi_1) \longrightarrow C^{\bullet}(\Phi_2) \longrightarrow \cdots$ is a double complex

# A van Est theorem

## Theorem

*Let $\mathcal{G}$ be a Lie 2-group with crossed module $G \longrightarrow H$. If $H$ is $k$-connected and $G$ is $(k-1)$-connected,*

$$H^n_{tot}(\Phi) = (0), \quad \forall n \leq k$$

## Proof.

$$\begin{array}{ccccccc}
C(H^\bullet, V) & \longrightarrow & C(\mathcal{G}_1^\bullet, V) & \longrightarrow & C(\mathcal{G}_2^\bullet, V) & \longrightarrow & \dots \\
\downarrow{\scriptstyle \Phi_0} & & \downarrow{\scriptstyle \Phi_1} & & \downarrow{\scriptstyle \Phi_2} & & \\
\bigwedge^\bullet \mathfrak{h}^* \otimes V & \longrightarrow & \bigwedge^\bullet \mathfrak{g}_1^* \otimes V & \longrightarrow & \bigwedge^\bullet \mathfrak{g}_2^* \otimes V & \longrightarrow & \dots
\end{array}$$

□

# A van Est theorem

## Theorem

*Let $\mathcal{G}$ be a Lie 2-group with crossed module $G \longrightarrow H$. If $H$ is $k$-connected and $G$ is $(k-1)$-connected,*

$$H^n_{tot}(\Phi) = (0), \quad \forall n \leq k$$

## Proof.

$$\begin{array}{ccccccc}
C^2(\Phi_0) & \longrightarrow & C^2(\Phi_1) & \longrightarrow & C^2(\Phi_2) & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
C^1(\Phi_0) & \longrightarrow & C^1(\Phi_1) & \longrightarrow & C^1(\Phi_2) & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
C^0(\Phi_0) & \longrightarrow & C^0(\Phi_1) & \longrightarrow & C^0(\Phi_2) & \longrightarrow & \dots
\end{array}$$

# A van Est theorem

## Theorem

*Let $\mathcal{G}$ be a Lie 2-group with crossed module $G \longrightarrow H$. If $H$ is $k$-connected and $G$ is $(k-1)$-connected,*

$$H^n_{tot}(\Phi) = (0), \quad \forall n \leq k$$

## Proof.

$$E_1^{p,q}: \quad \begin{array}{ccccccc} H^2(\Phi_0) & \longrightarrow & H^2(\Phi_1) & \longrightarrow & H^2(\Phi_2) & \longrightarrow & \ldots \\ H^1(\Phi_0) & \longrightarrow & H^1(\Phi_1) & \longrightarrow & H^1(\Phi_2) & \longrightarrow & \ldots \\ H^0(\Phi_0) & \longrightarrow & H^0(\Phi_1) & \longrightarrow & H^0(\Phi_2) & \longrightarrow & \ldots \end{array}$$

# An integrability result

$W \xrightarrow{\phi} V$ - 2-vector space

$\mathfrak{gl}(\phi)$:= The category of linear self functors and linear natural transformations

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- Fact: One can use the exponential to integrate linear Lie 2-algebras, i.e., linear subgroupoids of $\mathfrak{gl}(\phi)$

# An integrability result

$W \xrightarrow{\phi} V$ - 2-vector space
$\mathfrak{gl}(\phi)$:= The category of linear self functors and linear natural transformations

- Fact: One can use the exponential to integrate linear Lie 2-algebras, i.e., linear subgroupoids of $\mathfrak{gl}(\phi)$

## Theorem

*If* $\mathfrak{g}_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathfrak{h} \xrightarrow{u} \mathfrak{g}_1$ *is a Lie 2-algebra with*

$$\ker s \cap \mathfrak{c}(u(\mathfrak{h})) = (0),$$

*where* $\mathfrak{c}(u(\mathfrak{h}))$ *is the centralizer of* $u(\mathfrak{h})$ *in* $\mathfrak{g}_1$*, then* $\mathfrak{g}_1$ *is integrable*

# Why so hopeful?

**Theorem (Sheng,Zhu)**

*Finite-dimensional strict Lie 2-algebras are integrable*

# Why so hopeful?

## Theorem (Sheng,Zhu)

*Let $\mathcal{L} : \mathfrak{h} \longrightarrow \mathfrak{Der}(\mathfrak{g})$ be a Lie algebra action by derivations. Let $L : H \longrightarrow Aut(\mathfrak{g})$ be the unique group morphisms integrating $\mathcal{L}$. If $\zeta \in P(\mathfrak{h})$ is an $\mathfrak{h}$-homotopy class presenting $h \in H$.*

- *For $x \in \mathfrak{g}$, $L_h(x) = \xi(1)$, where $\xi \in P(\mathfrak{g})$ is the solution to:*

$$\frac{d}{d\lambda}\xi(\lambda) = \mathcal{L}_{\zeta(\lambda)}\xi(\lambda), \quad \xi(0) = x.$$

- *For $\xi \in P(\mathfrak{g})$, $\varphi_h\xi(\lambda) = \varpi(1, \lambda)$, where $\varpi(-, \lambda) \in P(\mathfrak{g})$ is the solution to:*

$$\frac{\partial}{\partial\lambda_0}\varpi(\lambda_0, \lambda_1) = \mathcal{L}_{\zeta(\lambda_0)}\varpi(\lambda_0, \lambda_1), \quad \varpi(0, \lambda) = \xi(\lambda).$$

*Thus, there is a group action $H \longrightarrow Aut(G)$*

$$[\xi]^h = [L_h \circ \xi] = [\varpi(1, -)].$$

# The data of a representation

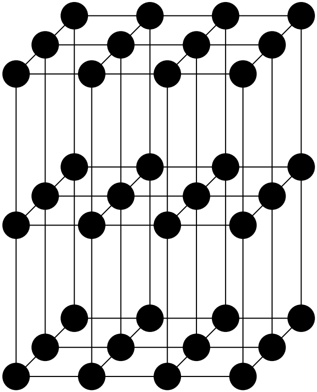
A Lie 2-algebra representation on $\phi : W \longrightarrow V$ consists of

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\rho_1} & \mathfrak{gl}(\phi)_1 = Hom(V,W) \\ {\scriptstyle\mu}\downarrow & & \downarrow \\ \mathfrak{h} & \xrightarrow[(\rho_0^0,\rho_0^1)]{} & \mathfrak{gl}(\phi)_0 \leq \mathfrak{gl}(V) \oplus \mathfrak{gl}(W) \end{array}$$

A Lie 2-group representation on $\phi : W \longrightarrow V$ consists of

$$\begin{array}{ccc} G & \xrightarrow{\rho_1} & GL(\phi)_1 \leq Hom(V,W) \\ {\scriptstyle i}\downarrow & & \downarrow \\ H & \xrightarrow[(\rho_0^0,\rho_0^1)]{} & GL(\phi)_0 \leq GL(V) \times GL(W) \end{array}$$

# Lie 2-algebras: The three dimensional grid

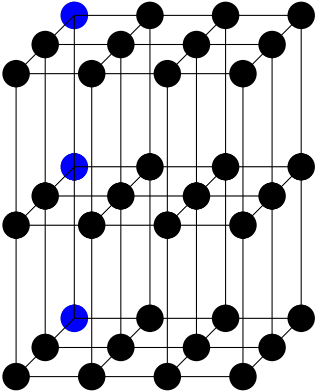


$$C_0^{p,q}(\mathfrak{g}_1, \phi) := \bigwedge^q \mathfrak{g}_p^* \otimes V$$

For $r > 0$,

$$C_r^{p,q}(\mathfrak{g}_1, \phi) := \bigwedge^q \mathfrak{g}_p^* \otimes \bigwedge^r \mathfrak{g}^* \otimes W$$

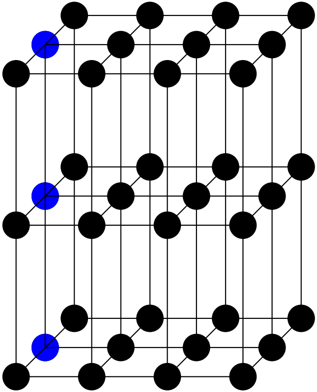
# Lie 2-algebras: The three dimensional grid



$$
\begin{array}{c}
\uparrow \\
\bigwedge^2 \mathfrak{h}^* \otimes V \\
\uparrow \\
\mathfrak{h}^* \otimes V \\
\uparrow \\
V
\end{array}
$$

The Chevalley-Eilenberg complex with respect to $\rho_0^0$

# Lie 2-algebras: The three dimensional grid



$$\uparrow$$

$$\bigwedge^2 \mathfrak{g}_1^* \otimes V$$

$$\uparrow$$

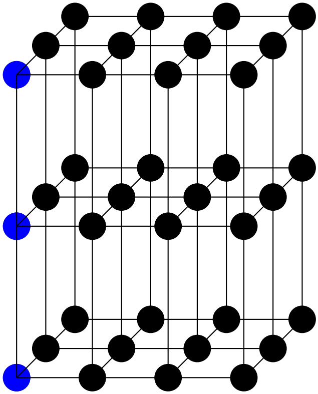
$$\mathfrak{g}_1^* \otimes V$$

$$\uparrow$$

$$V$$

The Chevalley-Eilenberg complex with respect to $\rho_0^0 \circ t$

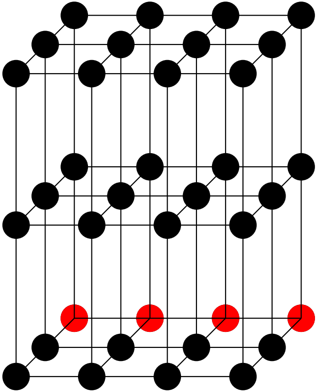
# Lie 2-algebras: The three dimensional grid



$$V \longrightarrow \mathfrak{g}_p^* \otimes V \longrightarrow \bigwedge^2 \mathfrak{g}_p^* \otimes V \longrightarrow$$

The Chevalley-Eilenberg complex with respect to $\rho_0^0 \circ t_p$

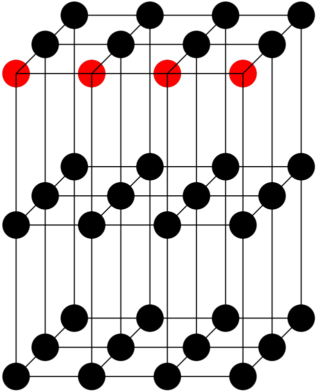
# Lie 2-algebras: The three dimensional grid



$$
V \xrightarrow{(\rho_1)_*} \mathfrak{g}^* \otimes W \longrightarrow \bigwedge^2 \mathfrak{g}^* \otimes W \longrightarrow
$$

The Chevalley-Eilenberg complex with respect to $\rho_0^1 \circ \mu$

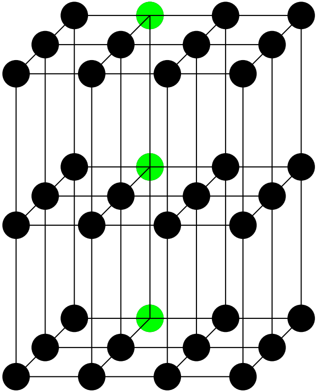
# Lie 2-algebras: The three dimensional grid



$$\begin{array}{c} \uparrow \\ \bigwedge^2 \mathfrak{g}^* \otimes (\bigwedge^q \mathfrak{g}_p^* \otimes W) \\ \uparrow \\ \mathfrak{g}^* \otimes (\bigwedge^q \mathfrak{g}_p^* \otimes W) \\ \uparrow {\scriptstyle (\rho_1)_*} \\ \bigwedge^q \mathfrak{g}_p^* \otimes V \end{array}$$

The Chevalley-Eilenberg complex with respect to $\rho_0^1 \circ \mu$

# Lie 2-algebras: The three dimensional grid



$$\uparrow$$

$$\bigwedge^2 \mathfrak{h}^* \otimes (\mathfrak{g}^* \otimes W)$$

$$\uparrow$$

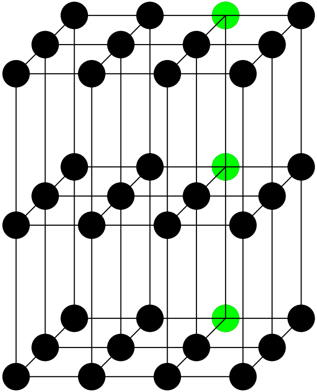
$$\mathfrak{h}^* \otimes (\mathfrak{g}^* \otimes W)$$

$$\uparrow$$

$$\mathfrak{g}^* \otimes W$$

The Chevalley-Eilenberg complex with respect to $\rho_0^1 - \mathcal{L}^*$

# Lie 2-algebras: The three dimensional grid



$$\bigwedge^2 \mathfrak{h}^* \otimes (\bigwedge^2 \mathfrak{g}^* \otimes W)$$

$$\uparrow$$

$$\mathfrak{h}^* \otimes (\bigwedge^2 \mathfrak{g}^* \otimes W)$$

$$\uparrow$$

$$\bigwedge^2 \mathfrak{g}^* \otimes W$$

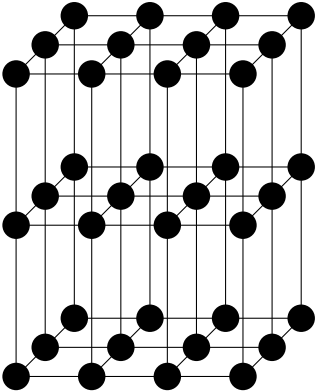
The Chevalley-Eilenberg complex with respect to $\rho^{(2)}$

# Lie 2-algebras: The three dimensional grid

In general, the representation of $\mathfrak{h}$ on $\bigwedge^r \mathfrak{g}^* \otimes W$ is given by the formula

$$\rho_y^{(r)}\omega(x_1, ..., x_r) = \rho_0^1(y)\omega(x_1, ..., x_r) - \sum_i \omega(x_1, ..., \mathcal{L}_y x_i, ..., x_r).$$

# Lie 2-groups: The three dimensional grid

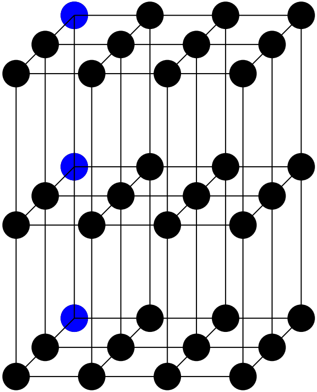


$$C_0^{p,q}(\mathcal{G},\phi) := C(\mathcal{G}_p^q, V)$$

For $r > 0$,

$$C_r^{p,q}(\mathcal{G},\phi) := C(\mathcal{G}_p^q \times G^r, W)$$

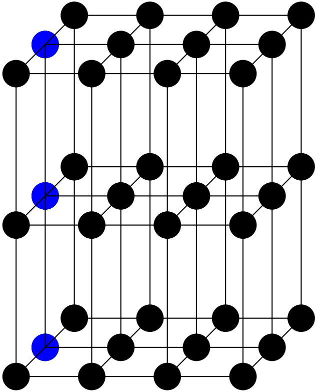
# Lie 2-groups: The three dimensional grid



$$V \longrightarrow C(H, V) \longrightarrow C(H^2, V) \longrightarrow$$

The group cochain complex with respect to $\rho_0^0$

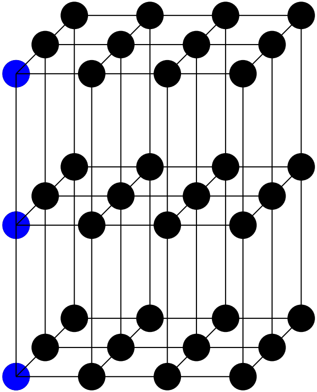
# Lie 2-groups: The three dimensional grid



$$\uparrow$$
$$C(\mathcal{G}^2, V)$$
$$\uparrow$$
$$C(\mathcal{G}, V)$$
$$\uparrow$$
$$V$$

The group cochain complex with respect to $\rho_0^0 \circ t$

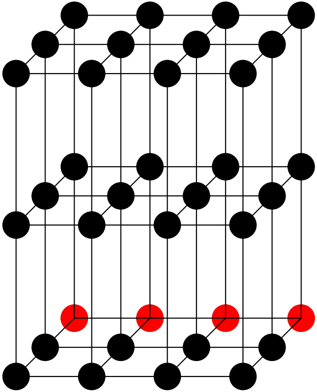
# Lie 2-groups: The three dimensional grid



$$V \longrightarrow C(\mathcal{G}_p, V) \longrightarrow C(\mathcal{G}_p^2, V) \longrightarrow$$

The group cochain complex with respect to $\rho_0^0 \circ t_p$

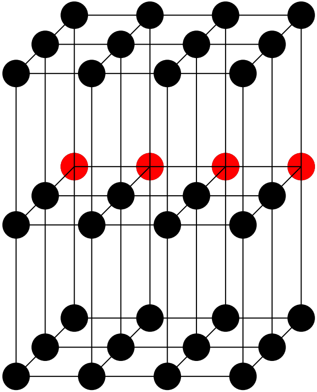
# Lie 2-groups: The three dimensional grid



$$V \xrightarrow{(\rho_1)_*} C(G, W) \longrightarrow C(G^2, W) \longrightarrow$$

The group cochain complex with respect to $\rho_0^1 \circ i$

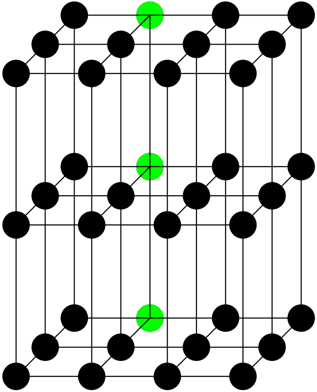
# Lie 2-groups: The three dimensional grid



$$\begin{array}{c} \uparrow \\ C(H \times G^2, W) \\ \uparrow \\ C(H \times G, W) \\ \uparrow (\rho_1)_* \\ C(H, V) \end{array}$$

The groupoid cochain complex of the Lie group bundle $H \times G \rightrightarrows H$

# Lie 2-groups: The three dimensional grid



$$\uparrow$$

$$C(H^2 \times G, W)$$

$$\uparrow$$

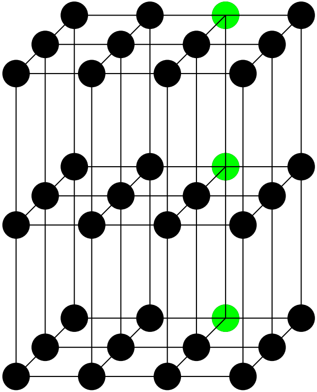
$$C(H \times G, W)$$

$$\uparrow$$

$$C(G, W)$$

The groupoid cochain complex of the action groupoid $G \ltimes H \rightrightarrows G$

# Lie 2-groups: The three dimensional grid



$$\begin{array}{c} \uparrow \\ C(H^2 \times G^2, W) \\ \uparrow \\ C(H \times G^2, W) \\ \uparrow \\ C(G^2, W) \end{array}$$

The groupoid cochain complex of the action groupoid $G^2 \rtimes H \rightrightarrows G^2$

# Lie 2-groups: The three dimensional grid

In general, the representation of the groupoids involved take values on trivial vector bundles.

- $p$-direction: $\mathcal{G}^q \times G^r \rightrightarrows H^q \times G^r$

$$(\vec{\gamma}; \vec{f}) \cdot (s(\vec{\gamma}); \vec{f}, w) := (t(\vec{\gamma}); \vec{f}, \rho_0^1(i(pr_G(\gamma_1 \bowtie \dots \bowtie \gamma_q)))^{-1} w)$$

- $q$-direction: $G^r \rtimes \mathcal{G}_p \rightrightarrows G^r$

$$(g_1, \dots, g_r; w) \cdot (\gamma; g_1, \dots, g_r) := (g_1^{t_p(\gamma)}, \dots, g_r^{t_p(\gamma)}; \rho_0^1(t_p(\gamma))^{-1} w)$$

- $r$-direction: $\mathcal{G}_p^q \times G \rightrightarrows \mathcal{G}_p^q$

$$(\gamma_1, \dots, \gamma_q; g) \cdot (\gamma_1, \dots, \gamma_q; w) := (\gamma_1, \dots, \gamma_q; \rho_0^1(i(g^{t_p(\gamma_1) \dots t_p(\gamma_q)}))w)$$

# Not triple complexes

- r=0:

$$\begin{array}{ccc} C_0^{p,q+1} & \xrightarrow{\partial} & C_0^{p+1,q+1} \\ \delta \uparrow & & \uparrow \delta \\ C_0^{p,q} & \xrightarrow[\partial]{} & C_0^{p+1,q} \end{array}$$

$(\delta\partial\omega)(\cdot) \bullet \longleftarrow \bullet (\partial\delta\omega)(\cdot)$ isomorphic in $V$

# Not triple complexes

- r=0:

$$\begin{array}{ccc} C_0^{p,q+1} & \xrightarrow{\partial} & C_0^{p+1,q+1} \\ \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\ C_0^{p,q} & \xrightarrow[\partial]{} & C_0^{p+1,q} \end{array}$$

$$(\delta\partial\omega)(\cdot) \bullet \longleftarrow \bullet (\partial\delta\omega)(\cdot)$$ isomorphic in $V$

- $r > 0$: $\delta \circ \partial$ and $\partial \circ \delta$ are homotopic as map of complexes

# The corrections

The complex of Lie 2-algebra cochains with values on $W \xrightarrow{\phi} V$

$$(C_{tot}(\bigwedge^q \mathfrak{g}_p^* \otimes \bigwedge^r \mathfrak{g}^* \otimes W), \nabla)$$

where

$$\nabla = \partial + (-1)^{p+q}(\delta + \delta_{(1)}) + \sum_{k=1}^{r} \Delta_k$$

and

$$\Delta_k : C_r^{p,q} \longrightarrow C_{r-k}^{p+1,q+k}$$

# The corrections

The complex of Lie 2-algebra cochains with values on $W \xrightarrow{\phi} V$

$$(C_{tot}(\bigwedge^{q} \mathfrak{g}_p^* \otimes \bigwedge^{r} \mathfrak{g}^* \otimes W), \nabla)$$

where

$$\nabla = \partial + (-1)^{p+q}(\delta + \delta_{(1)}) + \sum_{k=1}^{r} \Delta_k$$

For instance, for $k = 1$

$$\Delta_1 \omega \begin{pmatrix} x_0^0 & \cdots & x_q^0 \\ \vdots & \ddots & \vdots \\ x_0^p & \cdots & x_q^p \\ y_0 & \cdots & y_q \end{pmatrix} = \sum_{j=0}^{q} (-1)^j \omega( \begin{pmatrix} x_0^1 & \cdots & \hat{x}_j^1 & \cdots & x_q^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_0^p & \cdots & \hat{x}_j^p & \cdots & x_q^p \\ y_0 & \cdots & \hat{y}_j & \cdots & y_q \end{pmatrix} ; x_j^0)$$

# The corrections

The complex of Lie 2-algebra cochains with values on $W \xrightarrow{\phi} V$

$$(C_{tot}(\bigwedge^{q} \mathfrak{g}_p^* \otimes \bigwedge^{r} \mathfrak{g}^* \otimes W), \nabla)$$

where

$$\nabla = \partial + (-1)^{p+q}(\delta + \delta_{(1)}) + \sum_{k=1}^{r} \Delta_k$$

**Remark**

*This complex is isomorphic to the Chevalley-Eilenberg complex in the lowest degrees*

# The corrections

The complex of Lie 2-group cochains with values on $W \xrightarrow{\phi} V$

$$(C_{tot}(C(\mathcal{G}_p^q \times G^r, W), \nabla)$$

where

$$\nabla = (-1)^p(\delta_{(1)} + \sum_{a+b>0} (-1)^{(a+1)(r+b+1)} \Delta_{a,b})$$

and

$$\Delta_{a,b} : C_r^{p,q} \longrightarrow C_{r+1-(a+b)}^{p+a,q+b}$$

For instance, $\Delta_{1,0} = \partial$, $\Delta_{0,1} = \delta$ and...

# Another van Est theorem

$$\Phi : C_r^{p,q}(\mathcal{G},\phi) \longrightarrow C_r^{p,q}(\mathfrak{g}_1,\phi),$$

$$(\Phi\omega)(\xi_1,\ldots,\xi_q;z_1,\ldots,z_r) := \sum_{\sigma\in S_q}\sum_{\varrho\in S_r}|\sigma||\varrho|\overrightarrow{R}_{\sigma(\Xi)}\overrightarrow{R}_{\varrho(Z)}\omega,$$

where $\Xi=(\xi_1,\ldots,\xi_q)\in\mathfrak{g}_p^q$, $Z=(z_1,\ldots,z_r)\in\mathfrak{g}^r$, $|\cdot|$ stands for the sign of the permutation, and

$$(\overrightarrow{R}_{\varrho(Z)}\omega)(\vec{\gamma}) := \frac{d}{d\tau_r}|_{\tau_r=0}\cdots\frac{d}{d\tau_1}|_{\tau_1=0}\omega(\vec{\gamma};\exp_G(\tau_1 z_{\varrho(1)}),\ldots,\exp_G(\tau_r z_{\varrho(r)})), \quad \text{for } \vec{\gamma}\in\mathcal{G}_p^q;$$

$$\overrightarrow{R}_{\sigma(\Xi)}\overrightarrow{R}_{\varrho(Z)}\omega = \frac{d}{d\lambda_q}|_{\lambda_q=0}\cdots\frac{d}{d\lambda_1}|_{\lambda_1=0}(\overrightarrow{R}_{\varrho(Z)}\omega)(\exp_{\mathcal{G}_p}(\lambda_1\xi_{\sigma(1)}),\ldots,\exp_{\mathcal{G}_p}(\lambda_q\xi_{\sigma(q)})).$$

# Another van Est theorem

For constant $p$, $C(\mathcal{G}_p^\bullet \times G^\bullet, W)$ is the double complex associated to the double Lie groupoid

$$\begin{array}{ccc} \mathcal{G}_p \ltimes G & \rightrightarrows & \mathcal{G}_p \\ \downdownarrows & & \downdownarrows \\ G & \rightrightarrows & * \end{array}$$

Assembling column-wise groupoid van Est maps yields a map of double complexes to the double complex associated to its LA-groupoid

$$\begin{array}{ccc} \mathfrak{g}_p \ltimes G & \rightrightarrows & \mathfrak{g}_p \\ \downarrow & & \downarrow \\ G & \rightrightarrows & * \end{array}$$

# Another van Est theorem

$$C(\mathcal{G}_p^q \times G^r, W) \xrightarrow{\ \Phi\ } \bigwedge^q \mathfrak{g}_p^* \otimes \bigwedge^r \mathfrak{g}^* \otimes W,$$

$$C(\mathcal{G}_p^q \times G^r, W) \xrightarrow{\ \Phi_{col}\ } C(G^r, \bigwedge^q \mathfrak{g}_p^* \otimes W) \xrightarrow{\ \Phi_{row}\ } \bigwedge^q \mathfrak{g}_p^* \otimes \bigwedge^r \mathfrak{g}^* \otimes W$$

## Theorem

*If $\mathcal{G}$ is a Lie 2-group with crossed module $G \longrightarrow H$ and Lie 2-algebra $\mathfrak{g}_1$.*
*If both $G$ and $H$ are $k$-connected, then*

$$\Phi : H^n_\nabla(\mathcal{G}, \phi) \longrightarrow H^n_\nabla(\mathfrak{g}_1, \phi)$$

*is an isomorphism for $n \leq k$ and injective for $n = k + 1$.*

# The End

Thank you!