Reduction by local symmetries in Field theories

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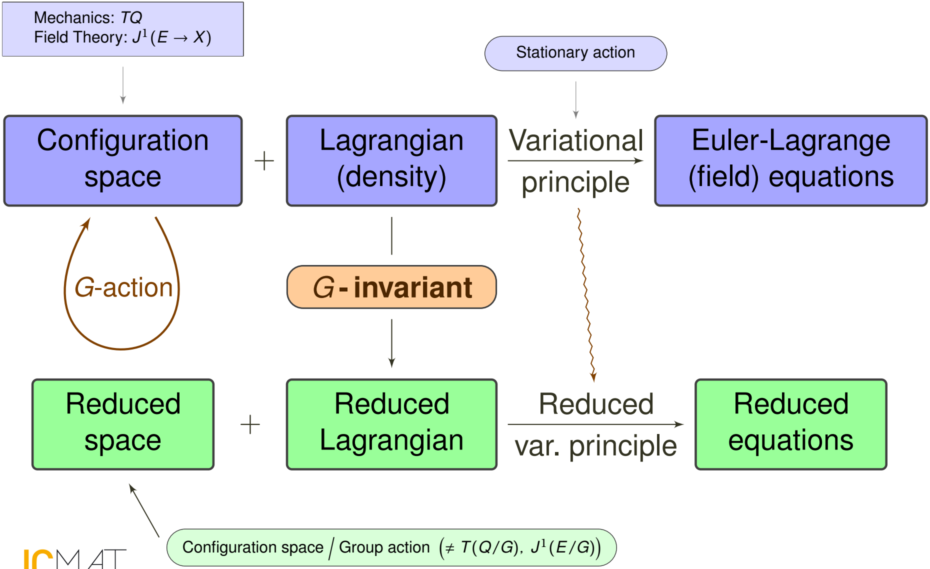
Mechanics: TQ
Field Theory: $J^1(E \to X)$
Stationary action
Configuration space
+
Lagrangian (density)
Variational principle
Euler-Lagrange (field) equations
G-action
G - invariant
Reduced space
+
Reduced Lagrangian
Reduced var. principle
Reduced equations
Configuration space / Group action $(\neq T(Q/G),\ J^1(E/G))$

Configuration. space	Group of symmetries	Reduced space / Reduced variations	Reduced equations
TG	G	$\mathfrak{g} = TG/G$ $\delta v = \dot\eta + [v,\eta]$	$\frac{d}{dt}\frac{\partial l}{\partial v} - \mathrm{ad}^*_v \frac{\partial l}{\partial v} = 0$ **Euler-Poincaré equations [2]**
TQ	G	$T(Q/G) \oplus_{Q/G} \mathrm{ad}(Q)$ $\delta^\omega \overline{v} = \frac{D\overline{\eta}}{Dt} + [\overline{v},\overline{\eta}]_G + \widetilde{B}(\delta x, \dot x)$	$\begin{cases} \frac{\partial l}{\partial x} - \frac{D}{Dt}\frac{\partial l}{\partial \dot x} = \left\langle \frac{\partial l}{\partial \overline{v}}, i_{\dot x}\widetilde{B} \right\rangle \text{ (Hor.)} \\ \frac{D}{Dt}\frac{\partial l}{\partial \overline{v}} - \mathrm{ad}^*_{\overline{v}} \frac{\partial l}{\partial \overline{v}} = 0 \text{ (Ver.)} \end{cases}$ **Lagrange-Poincaré equations [2]**
$J^1(P \longrightarrow X)$ $X = P/G$	G	$C(P) = J^1P/G$ $\delta^\omega s = \nabla^\omega \eta + [s^\omega, \eta]$	$\mathrm{div}^\omega \frac{\delta l}{\delta s} - \mathrm{ad}^*_{s^\omega} \frac{\delta l}{\delta s} = 0$ **Euler-Poincaré field equations [1]**
$J^1(E \longrightarrow X)$	G	$J^1(E/G) \oplus_{E/G} T^*X \otimes ad(E)$ $\delta^\omega \overline{s} = \nabla^\omega \overline{\eta} + [\overline{s}, \overline{\eta}] + \widetilde{B}(\delta\sigma, \sigma_*)$	$\begin{cases} \frac{\delta l}{\delta \sigma} - \mathrm{div}^{E/G} \frac{\delta l}{\delta j^1\sigma} = \left\langle \frac{\delta l}{\delta \overline{s}}, i_{\sigma_*}\widetilde{B} \right\rangle \text{ (Hor.)} \\ \mathrm{div}^\omega \frac{\delta l}{\delta \overline{s}} - \mathrm{ad}^*_{\overline{s}} \frac{\delta l}{\delta \overline{s}} = 0 \text{ (Ver.)} \end{cases}$ **Lagrange-Poincaré field equations [3]**

TABLE 1. PREVIOUS WORK ON REDUCTION IN LAGRANGIAN MECHANICS AND FIELD THEORY

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- Previous work: G does not depend on $x \in X$ (**Global symmetries**)
- What if it does? Family of actions: $E_x \times G_x \longrightarrow E_x$

Local symmetries

- **Lie group fiber bundle**: $G = \bigsqcup_{x \in X} G_x \longrightarrow X$
- **Fibered action**: $E \times_X G \longrightarrow E \quad \rightsquigarrow \quad J^1 E \times_X J^1 G \longrightarrow J^1 E$

PROBLEM: Reduction procedure for local symmetries

(i) Geometry of $J^1 E / J^1 G = \bigsqcup_{x \in X} J^1_x E / J^1_x G$?

(ii) Reduced variational principle?

(iii) Reduced eqs. for a G-invariant Lagrangian $L \colon J^1 E \longrightarrow \mathbb{R}$?

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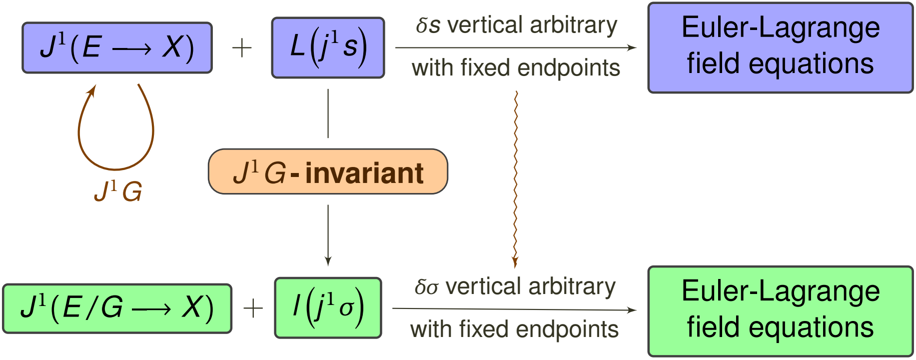
$J^1(E \longrightarrow X)$
+
$L(j^1 s)$
δs vertical arbitrary
with fixed endpoints
Euler-Lagrange field equations
$J^1 G$
$J^1 G$- invariant
$J^1(E/G \longrightarrow X)$
+
$l(j^1 \sigma)$
$\delta \sigma$ vertical arbitrary
with fixed endpoints
Euler-Lagrange field equations

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Lie group subbundle

Connection on $G \to X$ $\qquad$ $\mathfrak{h} \subset \mathfrak{g}$

$$\pi^*_{G,X}(T^*X \otimes \mathfrak{g}) \simeq J^1 G \supset H \simeq \pi^*_{G,X}(T^*X \otimes \mathfrak{h})$$

Generalized principal connection

A 1-form $\omega \in \Omega^1(E, \mathfrak{g})$ such that

- **Connection on** $E \to E/G$: $\omega(\xi^*) = \xi \quad \forall \xi \in \mathfrak{g}$
- **Equivariant**: $\forall (U_y, U_g) \in T_y E \times_{T_x X} T_g G$

$$\omega_{y \cdot g}\left((d\Phi)_{(y,g)}(U_y, U_g)\right) = Ad_{g^{-1}}\left(\omega_y(U_y) + \overline{\upsilon}(U_g)\right)$$

Remark: $\Phi : E \times_X G \longrightarrow E$

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$J^1(E \longrightarrow X)$ + $L\left(j^1 s\right)$ —— δs vertical arbitrary ⟶ **Euler-Lagrange field equations**

$H \simeq \pi^*_{G,X}\left(T^*X \otimes \mathfrak{h}\right)$

***H*-invariant**

$J^1(E/G) \oplus_{E/G} T^*X \otimes ad_{\mathfrak{h}}(E)$ + $l\left(j^1\sigma, \overline{s}\right)$ ⟶

$$\begin{cases} \frac{\delta l}{\delta \sigma} - \mathrm{div}^{E/G}\left(\frac{\delta l}{\delta j^1\sigma}\right) = \left\langle \frac{\delta l}{\delta \overline{s}}, i_{\sigma_*}\widetilde{B}\right\rangle \\ \mathrm{div}^{\mathfrak{h}}\left(\frac{\delta l}{\delta \overline{s}}\right) - \mathrm{ad}^*\left(\overline{s}\right)\left(\frac{\delta l}{\delta \overline{s}}\right) = 0 \end{cases}$$

$$\mathrm{ad}_{\mathfrak{h}}(E) = \frac{E\times_X \mathfrak{g}}{G\times_X \mathfrak{h}} \longrightarrow E/G$$

$$\begin{cases} \delta\sigma \text{ vertical arbitrary} \\ \delta^{\omega}\overline{s} = \overline{\nabla}^{\mathfrak{h}}\overline{\eta} - \left[\overline{s}, \overline{\eta}\right] + \widetilde{B}\left(\delta\sigma, \sigma_*\right) \end{cases}$$

Remarks:

- $\overline{\eta} \in \Gamma\left(ad_{\mathfrak{h}}(E) \longrightarrow X\right)$
- ω yields a connection $\nabla^{\mathfrak{h}}$ on $ad_{\mathfrak{h}}(E) \to E/G$

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- General Lie group subbundles: $J^1 G \supset H \neq \pi^*_{G,X}\left(T^* X \otimes \mathfrak{h}\right)$
- Reconstruction:

$s \in \Gamma\left(J^1 E \to X\right)$ satisfying Euler-Lagrange equations

Reduction ↓ ↑ Reconstruction

$\overline{s} \in \Gamma\left(T^* X \otimes ad_{\mathfrak{h}}(E) \to X\right)$ satisfying reduced equations

- Examples and applications: Gauge Theories

Yang-Mills (electromagnetism, weak force, chromodynamics)

- $E = C \to X$ of a principal $\mathcal{G}$-bundle $P \to X$
- $G = J^1 Ad(P) \to X$, with $Ad(P) = \left(P \times \mathcal{G}\right)/\mathcal{G}$
- Locally: $A_x \cdot j^1_x g = (dg)_x \, g(x)^{-1} + g(x) \, A_x \, g(x)^{-1}$

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THANK YOU!!!

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