Abelianisation of Meromorphic $\mathrm{GL}(2,\mathbb{C})$-Connections

based on `arXiv:1902.03384` and work in progress

Nikita Nikolaev

The University Of Sheffield.

19 March 2021

Consider:

- X $:=$ a Riemann surface (compact)
- $\mathcal{E}$ $:=$ a holomorphic vector bundle on X
  $=$ sheaf of holomorphic sections of a holomorphic vector bundle on X
- D $\subset$ X $:=$ effective divisor $:=$ discrete points with positive multiplicity
- A ***meromorphic connection*** on $\mathcal{E}$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \omega_{\mathsf{X}}(\mathsf{D})$$

satisfying the Leibniz rule: for any local section $e \in \mathcal{E}$ and holomorphic function f,

$$\nabla(fe) = f\nabla(e) + e \otimes \mathrm{d}f.$$

- Locally, $\nabla = \mathrm{d} + \phi$ where $\phi =$ endomorphism of $\mathcal{E}$ with values in $\omega_{\mathsf{X}}(\mathsf{D})$
- If p $\in$ D has multiplicity $m \geqslant 1$ and $z(\mathsf{p}) = 0$, then $\nabla = \mathrm{d} + A(z) z^{-m}\, \mathrm{d}z$
  where $A(z) =$ holomorphic matrix
- Locally, the same as a singular ODE $\nabla_{\partial_z} e(z) = \partial_z e(z) + A(z) z^{-m} e(z) = 0.$

- Observation: if we plug in not the vector field ∂_z but $z^m \partial_z$, then the covariant derivative $\nabla_{z^m \partial_z}$ is a $\mathbb{C}$-linear map $\mathcal{E} \to \mathcal{E}$.
- Vector fields of the form $z^m \partial_z$ form a rank-one Lie algebroid

$$\mathcal{A}_{\mathsf{X}} := \mathcal{T}_{\mathsf{X}}(-\mathsf{D}) \hookrightarrow \mathcal{T}_{\mathsf{X}}$$

  of holomorphic vector fields vanishing along D.
- Fact: Any Lie algebroid of rank one on a curve X is either a bundle of abelian Lie algebras or it is of the form $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$ for some divisor $\mathsf{D} \subset \mathsf{X}$.
- $\dim \mathsf{X} = 1 \quad \implies \quad$ no curvature $\Leftrightarrow \nabla_{[u,v]} = \nabla_u \nabla_v - \nabla_v \nabla_u$
- $\Rightarrow (\mathcal{E}, \nabla) \in \mathrm{Rep}(\mathcal{A}_{\mathsf{X}})$ is a ***representation*** of the Lie algebroid $\mathcal{A}_{\mathsf{X}}$.

- $\mathcal{A}_{\mathsf{X}} = \mathcal{T}_{\mathsf{X}}(-\mathsf{D})$ has a ssc integration $\Pi_1(\mathsf{X}, \mathsf{D}) =$ ***twisted fundamental groupoid***
- Lie algebroid representation $(\mathcal{E}, \nabla)$ integrates to Lie groupoid representation $(\mathcal{E}, \Psi)$ where $\Psi : \Pi_1(\mathsf{X}, \mathsf{D}) \longrightarrow \mathsf{GL}(\mathcal{E})$ is universal parallel transport operator for ∇.

Pushforward of Connections Along Branched Covers

- Let $\pi : \mathsf{Y} \to \mathsf{X}$ be branched $n : 1$ cover
- Let $\mathsf{B} \subset \mathsf{X}$, $\mathsf{R} \subset \mathsf{Y}$ are branch and ramification loci (assume simple)
- $\mathsf{C} := \pi^* \mathsf{D} \subset \mathsf{Y}$ or $\mathsf{C} := \pi^* \mathsf{D} \cup \mathsf{R} \subset \mathsf{Y}$ divisor, $\mathcal{A}_\mathsf{Y} := \mathcal{T}_\mathsf{Y}(-\mathsf{C})$
- Pushforward $\pi_* : \mathrm{Rep}(\mathcal{A}_\mathsf{Y}) \longrightarrow \mathrm{Rep}(\mathcal{A}'_\mathsf{X})$ where $\mathcal{A}'_\mathsf{X} := \mathcal{A}_\mathsf{X}(-\mathsf{B}) = \mathcal{T}_\mathsf{X}\big(-(\mathsf{D} \cup \mathsf{B})\big)$.
- In particular, rank-1 representations push down to rank-n representations:

$$\pi_* : \mathrm{Rep}^1(\mathcal{A}_\mathsf{Y}) \longrightarrow \mathrm{Rep}^n(\mathcal{A}'_\mathsf{X})$$

- **Question:** can every rank-n representation be seen as the pushforward of some rank-1 representation?
  Answer: absolutely not! $\pi_* \partial$ has very special quasi-permutation monodromy around B corresponding to π.
- **Our Goal:** given correct assumptions on Y, build an equivalence

$$\mathrm{Rep}^1_*(\mathcal{A}_\mathsf{Y}) \simeq \mathrm{Rep}^n_\Gamma(\mathcal{A}_\mathsf{X})$$

  - $\mathrm{Rep}^1_*(\mathcal{A}_\mathsf{Y}) \subset \mathrm{Rep}^1(\mathcal{A}_\mathsf{Y})$ obtained by fixing residues along R
  - $\mathrm{Rep}^n_\Gamma(\mathcal{A}_\mathsf{X}) \subset \mathrm{Rep}^n(\mathcal{A}_\mathsf{X})$ obtained by genericity wrt chosen combinatorial data Γ on X

Theorem (N)

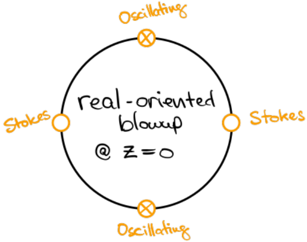
*For $n = 2$, there is indeed such an equivalence, called **abelianisation**.*

Isotropy Representations and Stokes Sectors

- If $\mathsf{p} \in \mathsf{D}$ with multiplicity m, get ***isotropy Lie algebra***:

$$\mathfrak{iso}_{\mathsf{p}}(\mathcal{A}_{\mathsf{X}}) := \ker\left(\mathcal{A}_{\mathsf{X}}|_{\mathsf{p}} \longrightarrow \mathcal{T}_{\mathsf{X}}|_{\mathsf{p}} \right) \cong (\mathsf{T}^*_{\mathsf{p}}\mathsf{X})^{m-1}$$

- Given a representation $(\mathcal{E}, \nabla) \in \mathrm{Rep}(\mathcal{A}_{\mathsf{X}})$ and $\mathsf{p} \in \mathsf{D}$, get ***isotropy representation*** $\mathfrak{iso}_{\mathsf{p}}(\nabla) : \mathfrak{iso}_{\mathsf{p}}(\mathcal{A}_{\mathsf{X}}) \to \mathsf{End}(\mathcal{E}|_{\mathsf{p}})$
- If $m = 1$, $\mathfrak{iso}_{\mathsf{p}}(\nabla)$ is just the residue matrix $A(0)$ of ∇ at p. If $m \geqslant 2$, $\mathfrak{iso}_{\mathsf{p}}(\nabla)$ is the leading term of the principal part $A(0)$ of ∇ at p.
- eigenvalues of $\mathfrak{iso}_{\mathsf{p}}(\nabla) \in \mathsf{End}(\mathcal{E}|_{\mathsf{p}}) \otimes (\mathsf{T}_{\mathsf{p}}\mathsf{X})^{m-1}$ are elements $\lambda_1, \ldots, \lambda_n \in (\mathsf{T}_{\mathsf{p}}\mathsf{X})^{m-1}$ $\Rightarrow$ weights $\lambda_{ij} := \lambda_i - \lambda_j$ for the adjoint action on $\mathsf{End}(\mathcal{E}|_{\mathsf{p}})$
- For $m \geqslant 2$, $v \in \mathsf{T}_{\mathsf{p}}\mathsf{X}$ is an ***(ij)-Stokes vector*** if $v^{m-1} = \lambda_{ij}$.
- Let $\mathsf{D}_{\mathrm{irreg}} \subset \mathsf{D}$ points with multiplicity $\geqslant 2$ (irregular locus). Let $\widetilde{\mathsf{X}} :=$ real-oriented blowup of X along $\mathsf{D}_{\mathrm{irreg}}$. Let $\widetilde{\mathsf{D}}_{\mathrm{irreg}} :=$ preimage of $\mathsf{D}_{\mathrm{irreg}}$ $=$ disjoint union of circles $\mathbb{S}^1$.



- Assume: ∇ has ***generic polar data*** $:=$ $\mathfrak{iso}_{\mathsf{p}}(\nabla)$ has distinct real parts (if $m = 1$) or distinct eigenvalues (if $m \geqslant 2$). Then the number of Stokes and oscillating directions is maximal: for $n = 2$, there are $2(m-1)$ directions of each kind.
- In each Stokes sector (for $m \geqslant 2$) or in any sector near a simple pole (for $m = 1$), get locally-defined flat ***Levelt filtrations***:

$$\mathcal{E}^\bullet = \left(\mathcal{E}^1 \subset \mathcal{E}^2 \subset \cdots \subset \mathcal{E}\right)$$

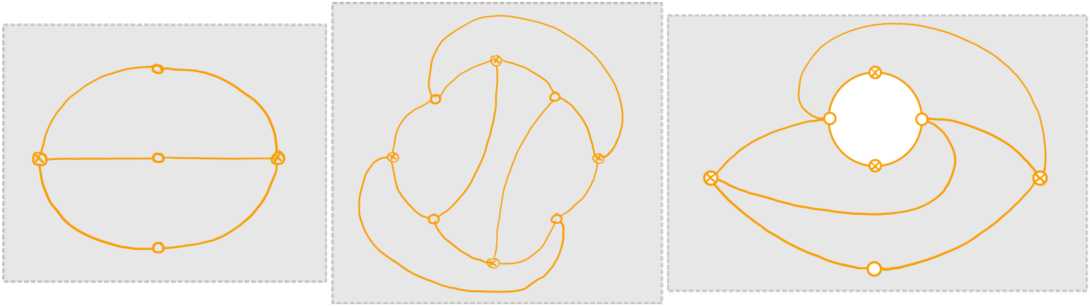
  by growth rates of sections as they are parallel transported into the Stokes direction.
- Get ***locally filtered representations***: near each simple pole or Stokes direction,

$$(\mathcal{E}, \nabla) \cong (\mathcal{E}^\bullet, \nabla) \qquad \text{where} \qquad \nabla : \mathcal{E}^k \to \mathcal{E}^k \otimes \omega_{\mathsf{X}}(\mathsf{D})$$
$$(\mathcal{E}, \Psi) \cong (\mathcal{E}^\bullet, \Psi) \qquad \text{where} \qquad \Psi : \mathsf{G} \longrightarrow \mathsf{GL}(\mathcal{E}^\bullet) .$$

- If ∇ has generic polar data, each $\mathcal{E}^\bullet$ is a full filtration.

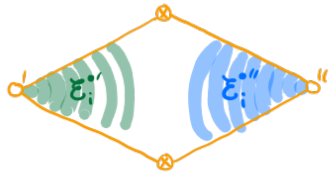
Consider:

- $\pi : \mathsf{Y} \to \mathsf{X}$ ramified double cover; $\mathsf{B} \subset \mathsf{X}$ branch locus, $\mathsf{R} \subset \mathsf{Y}$ ramification locus
- Assume: $\mathsf{B} \cap \mathsf{D} = \varnothing$ and genus $g_{\mathsf{Y}} = |\mathsf{D}| + 4g_{\mathsf{X}} - 3$.
- Introduce two colours $\otimes, \bigcirc$ for points in B and D as follows:
  - $\otimes$ for each point in B
  - $\bigcirc$ for each point in D_{reg}
  - $\bigcirc$ for each Stokes direction in $\widetilde{\mathsf{D}}_{\text{irreg}}$
  - $\otimes$ for each oscillating direction in $\widetilde{\mathsf{D}}_{\text{irreg}}$
- **Definition**: A (simple, saddle-free) ***Stokes graph*** Γ on (X, D) adapted to π is a bipartite squaregraph on X with vertex colours $\big(\{\otimes\}, \{\bigcirc\}\big)$ which is
  1. trivalent at each $\otimes \in \mathsf{B}$;
  2. bivalent at each $\otimes \in \widetilde{\mathsf{D}}_{\text{irreg}}$ with the two edges being the circle boundary arcs.



Genericity with respect to Γ

- On each face U_i of Γ, $(\mathcal{E}, \nabla)$ is filtered in two ways: $\mathcal{E}_i^{\bullet\prime}, \mathcal{E}_i^{\bullet\prime\prime}$ coming from poles $\bigcirc', \bigcirc''$. We say the Levelt filtrations of $(\mathcal{E}, \nabla)$ are ***generic wrt* $\boldsymbol{\Gamma}$** if $\mathcal{E}_i^{\bullet\prime} \pitchfork \mathcal{E}_i^{\bullet\prime\prime}$ for face U_i.

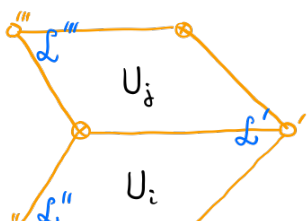


- **Key property 1**: if the Levelt filtrations of $(\mathcal{E}, \nabla)$ are generic wrt Γ, then we get canonical flat decompositions over each face U_i:

$$\varphi_i : \mathcal{E}_i \xrightarrow{\sim} \mathcal{L}_i' \oplus \mathcal{L}_i''$$

- **Key property 2**: Given two adjacent faces $\mathsf{U}_i, \mathsf{U}_j$, over the (ij) edge, get a filtered flat **decomposition-comparison isomorphism**:

$$\varphi_{ij} := \varphi_j \circ \varphi_i^{-1} = \begin{bmatrix} 1 & \Delta_{ij} \\ 0 & g_{ij} \end{bmatrix} : \begin{array}{ccc} \mathcal{L}_{ij}' & \xrightarrow{\;1\;} & \mathcal{L}_{ij}' \\ \oplus & \nearrow & \oplus \\ \mathcal{L}_{ij}'' & \xrightarrow[g_{ij}]{} & \mathcal{L}_{ij}''' \end{array}$$



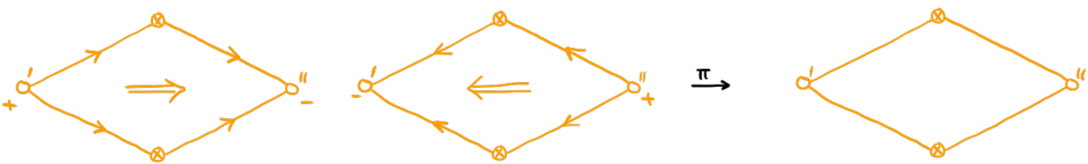
- Let $\mathrm{Rep}^2_\Gamma(\mathcal{A}_\mathsf{X}) :=$ category of rank-two representations of $\mathcal{A}_\mathsf{X} = \mathcal{T}_\mathsf{X}(-\mathsf{D})$ whose Levelt filtrations are generic wrt Γ.
- Let $\mathrm{Rep}^1_*(\mathcal{A}_\mathsf{Y}) :=$ category of rank-one representations of $\mathcal{A}_\mathsf{Y} = \mathcal{T}_\mathsf{Y}(-\pi^*\mathsf{D} - \mathsf{R})$ which have residues $-1/2$ at ramification points.

Theorem (N)

There is an equivalence of categories

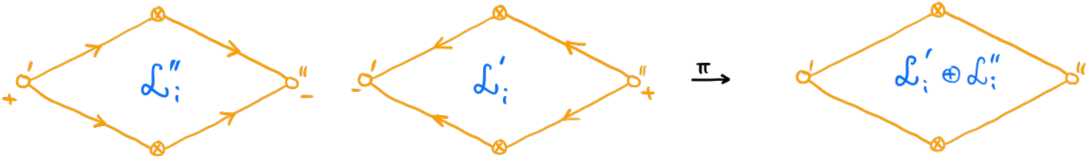
$$\pi^{\mathrm{ab}}_\Gamma : \mathrm{Rep}^2_\Gamma(\mathcal{A}_\mathsf{X}) \xrightarrow{\sim} \mathrm{Rep}^1_*(\mathcal{A}_\mathsf{Y})$$
$$(\mathcal{E}, \nabla) \longmapsto (\mathcal{L}, \partial) \ .$$

- The inverse equivalence π^Γ_{ab} is a local deformation of π_*.
- π^{ab}_Γ depends on the choice of a lift of Γ to a well-oriented double cover graph $\vec{\Gamma}$ on Y:



- Only two possible such choices of $\vec{\Gamma}$ related by the canonical involution $\sigma : \mathsf{Y} \to \mathsf{Y}$, and the two choices of π^{ab}_Γ are intertwined by σ^*.

Constructing $\pi_\Gamma^{\mathrm{ab}} : (\mathcal{E}, \nabla) \mapsto (\mathcal{L}, \partial)$ — Main Idea



- Each face U_i with polar vertices $\bigcirc', \bigcirc''$ lifts to two faces $\mathsf{U}_i', \mathsf{U}_i''$ of $\vec{\Gamma}$
- Lift $\mathcal{L}_i'$ to U_i' and $\mathcal{L}_i''$ to U_i''.
- Glue $\mathcal{L}_i''$ to $\mathcal{L}_j'''$ by g_{ij} = diagonal entry of the decomposition-comparison isomorphism:

$$\varphi_{ij} := \varphi_j \circ \varphi_i^{-1} = \begin{bmatrix} 1 & \Delta_{ij} \\ 0 & g_{ij} \end{bmatrix} : \begin{array}{ccc} \mathcal{L}_{ij}' & \xrightarrow{\;1\;} & \mathcal{L}_{ij}' \\ \oplus & \nearrow & \oplus \\ \mathcal{L}_{ij}'' & \xrightarrow[g_{ij}]{} & \mathcal{L}_{ij}''' \end{array} .$$

- The remaining off-diagonal information Δ_{ij} is used to invert π_Γ^{ab}.

Constructing $\pi^{\Gamma}_{\mathrm{ab}} : (\mathcal{L}, \partial) \mapsto (\mathcal{E}, \nabla)$ — Local Groupoid Cocycle

- Strategy: given $(\mathcal{E}, \nabla)$, construct $(\mathcal{L}, \partial)$, then compare $(\mathcal{E}, \nabla)$ with $(\pi_* \mathcal{L}, \pi_* \partial)$.
- On each face U_i, by construction of $\mathcal{L}$, get canonical isomorphisms

$$\varphi_i : \mathcal{E}_i \xrightarrow{\sim} \mathcal{L}'_i \oplus \mathcal{L}''_i = \pi_* \mathcal{L}_i$$

- Over each edge (ij), interpret decomposition-comparison isomorphisms $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$ as automorphisms of $\pi_* \mathcal{L}$:

$$\varphi_{ij} = \begin{bmatrix} 1 & \Delta_{ij} \\ 0 & 1 \end{bmatrix} \in \mathsf{Aut}\left(\pi_* \mathcal{L}_{ij}\right) .$$

- Each φ_{ij} is a Čech-groupoid 1-cocycle with values in the representation $\mathsf{Aut}(\pi_* \mathcal{L})$:
  Let $\mathsf{G_X} := \Pi_1(\mathsf{X}, \mathsf{B} \cup \mathsf{D})$ and $\mathsf{G_Y} := \Pi_1(\mathsf{Y}, \mathsf{R} \cup \pi^* \mathsf{D})$ be the relevant groupoids.
  Let $(\mathcal{E}, \Psi), (\mathcal{L}, \psi), (\pi_* \mathcal{L}, \pi_* \psi)$ be the corresponding representations of $\mathsf{G_X}$ and $\mathsf{G_Y}$.
  Over each U_i, use φ_i to transport Ψ to representation Φ on $\pi_* \mathcal{L}$: $\quad \Phi_i = \varphi_i \Psi_i \varphi_i^{-1}$.
  Then:

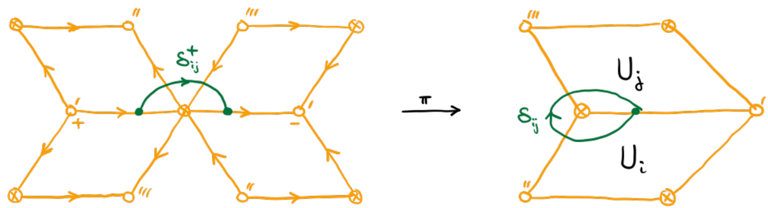
$$\varphi_{ij} = \Phi_i \circ (\pi_* \psi_{ij})^{-1} \in \mathsf{Z}^1\left(\mathsf{G}_{ij}, \mathsf{Aut}(\pi_* \mathcal{L}_{ij})\right)$$

  where $\mathsf{G}_{ij} :=$ identity-connected component of $\mathsf{G}|_{\mathsf{U}_{ij}}$.
- φ_{ij} restricts to id on $\bigcirc \qquad \leftrightarrow \qquad \pi_* \mathcal{L}_{\mathsf{p}} \xrightarrow{\sim} \mathsf{gr}(\mathcal{E}^{\bullet}_{\mathsf{p}})$ for each $\mathsf{p} \in \mathsf{D}$
- φ_{ij} restricts to $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ on $\otimes \in \mathsf{B} \quad \leftrightarrow \quad [\pi] \simeq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Constructing $\pi_{\mathrm{ab}}^{\Gamma} : (\mathcal{L}, \partial) \mapsto (\mathcal{E}, \nabla)$ — Main Idea

- For each edge (ij), look again at the formula $\varphi_{ij} = \begin{bmatrix} 1 & \Delta_{ij} \\ 0 & 1 \end{bmatrix} \in \mathsf{Aut}\left(\pi_* \mathcal{L}_{ij}\right)$.
- **Crucial observation** [essentially by Gaiotto-Moore-Neitzke]: Can interpret Δ_{ij} as the parallel transport of ∂ along a δ_{ij}^{+}:

$$\Delta_{ij} = \mathrm{Par}(\partial, \delta_{ij}^{+})$$



- This ***path-lifting rule*** does not depend on $(\mathcal{L}, \partial)$, so we get a Čech-groupoid 1-cocycle with values in the sheaf $\mathcal{A}ut(\pi_*)$ of natural automorphisms of π_*:

$$\widehat{\varphi} := \mathrm{id} + \widehat{\Delta} \in \check{\mathsf{Z}}^1\big(\mathsf{G}_{\mathsf{X}}, \mathcal{A}ut(\pi_*)\big) \qquad \text{where} \qquad \widehat{\Delta} = \left\{ \widehat{\Delta}_{ij} = \mathrm{Par}(-, \delta_{ij}^{+}) \right\}$$

- Finally, $\pi_{\mathrm{ab}}^{\Gamma} := \widehat{\varphi} \cdot \pi_*$

Thank you for your attention!