Hyperkähler realizations of holomorphic Poisson surfaces

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Definition (Calabi 1979)

A **hyperkähler manifold** is a Riemannian manifold (M, g) with three complex structures I, J, K that are Kähler with respect to g and satisfy the quaternionic identities $I^2 = J^2 = K^2 = IJK = -1$.

Hyperkähler $\implies$ holomorphic symplectic

- 3 real symplectic forms: $\omega_I = g(I\cdot, \cdot)$, $\omega_J = g(J\cdot, \cdot)$, $\omega_K = g(K\cdot, \cdot)$.
- $\Omega = \omega_J + i\omega_K$ is holomorphic symplectic with respect to I.

Converse. *Compact* holomorphic symplectic and Kähler $\implies$ hyperkähler
(No such general result for non-compact manifolds.)

Symplectic realizations. Every holomorphic Poisson manifold X integrates to a holomorphic symplectic local groupoid $M \rightrightarrows X$. Is M hyperkähler on a neighbourhood of the identity section?

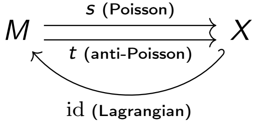
This talk. Yes, if X is compact Kähler and $\dim_{\mathbb{C}} X = 2$.

Proof. Lift special deformations of Poisson structures to build a twistor space.

e.g. Zero Poisson structure $\rightarrow$ Feix-Kaledin hyperkähler structure on T^*X.

Holomorphic symplectic realizations

- A **holomorphic Poisson manifold** is a complex manifold X together with a holomorphic bivector field π such that $\{f, g\} := \pi(df, dg)$ is a Lie bracket on $\mathcal{O}_X$.
- A (strict) **symplectic realization** of (X, π) is a holomorphic symplectic manifold (M, Ω) together with a surjective holomorphic Poisson submersion $s : (M, \Omega) \to (X, \pi)$ and a Lagrangian section $X \hookrightarrow M$.
- **Holomorphic Karasev-Weinstein theorem.** *Every holomorphic Poisson manifold has a symplectic realization. Proofs.* Laurent-Gengoux, Stiénon, Xu: Integration of holomorphic Poisson is equivalent to integration of the underlying real Poisson. Broka–Xu: Holomorphic Crainic-Mărcuț formula.
- The restriction of M to a neighbourhood of X is still a symplectic realization, and all symplectic realizations are isomorphic near X.
- After restricting to a neighbourhood of X, M has a unique structure of a **holomorphic symplectic local groupoid**:



  - $M \rightrightarrows X$ holomorphic local Lie groupoid
  - Ω holomorphic symplectic form on M
  - $\Gamma_{\mathrm{mult}} \subseteq M \times M \times M^-$ complex Lagrangian

X complex manifold, $\pi = 0$

Symplectic realization.

- $M = T^*X$ with canonical holomorphic symplectic form Ω_{can}
- $s : T^*X \to X$ bundle map
- $\mathrm{id} : X \hookrightarrow T^*X$ zero section

Theorem (Feix 1999, Kaledin 1999)

For any real-analytic Kähler form ω on X, there is a unique hyperkähler structure (g, I, J, K) on a neighbourhood of the zero section of T^*X such that $\omega_I|_X = \omega$ and $\omega_J + i\omega_K = \Omega_{can}$.

Proof (Feix). Twistor theory. □

Remarks.

- The hyperkähler structure might not extend to all of T^*X.
  (e.g. if X is a Riemann surface of constant negative curvature.)
- Every Kähler class contains a real-analytic representative.

Conclusion. If X is Kähler then $(X, \pi = 0)$ has a hyperkähler realization.

(X, Ω) holomorphic symplectic manifold, $\pi = \Omega^{-1}$

Symplectic realization.

- $M = X \times X$ (pair groupoid), with $(\Omega, -\Omega)$.
- $s : X \times X \to X$, $s(x, y) = x$
- $\mathrm{id} : X \hookrightarrow X \times X$, $\mathrm{id}(x) = (x, x)$.

Theorem (Beauville 1983)

If (X, Ω) is a compact holomorphic symplectic Kähler manifold, then there is a hyperkähler structure (g, I, J, K) on X such that $\Omega = \omega_J + i\omega_K$.

Proof. Yau's solution to the Calabi conjecture + Bochner's principle □

Conclusion. If (X, Ω) is holomorphic symplectic and is compact Kähler then $(X, \pi = \Omega^{-1})$ has a hyperkähler realization.

$X = \mathfrak{g}^*$, where $\mathfrak{g}$ is a complex semisimple Lie algebra

Symplectic realization.

- T^*G, where $\mathrm{Lie}(G) = \mathfrak{g}$.
- $s : T^*G = G \times \mathfrak{g}^* \to \mathfrak{g}^* : (g, \xi) \mapsto \xi$
- $\mathrm{id} : \mathfrak{g}^* = T^*_1 G \hookrightarrow T^*G$.

Theorem (Kronheimer 1988)

There is a complete hyperkähler structure on T^*G.

Proof. $G = K_{\mathbb{C}}$, K compact. Infinite-dimensional hyperkähler quotient:

$$\mathcal{M} = \underset{\text{(the one-dimensional reduction of the anti-self-dual Yang–Mills equations)}}{\text{moduli space of solutions to Nahm's equations on } [0,1]}$$

$$= \left\{(\alpha,\beta) : [0,1] \to \mathfrak{g} \times \mathfrak{g} : {\scriptstyle \dot{\beta}+[\alpha,\beta]=0 \atop \dot{\alpha}+\dot{\alpha}^*+[\alpha,\alpha^*]+[\beta,\beta^*]=0}\right\} / C^\infty([0,1], K)_0.$$

$$= \left\{(\alpha,\beta) : [0,1] \to \mathfrak{g} \times \mathfrak{g} : \dot{\beta} + [\alpha,\beta] = 0\right\} / C^\infty([0,1], G)_0 \qquad (\star)$$

$$= T^*G \qquad \square$$

$(\star)$ is the Cattaneo–Felder construction of the groupoid of $\mathfrak{g}^*$ as an infinite-dimensional symplectic reduction of the space of cotangent paths.

Holomorphic Poisson structures with hyperkähler realizations:

- **Zero.** *Proof.* Twistor theory.
- **Non-degenerate.** *Proof.* Calabi conjecture.
- **Kirillov–Kostant–Souriau.** *Proof.* Gauge theory.

Question. *Does every holomorphic Poisson Kähler manifold X have a hyperkähler realization $M \to X$?*

This talk. Yes, if X is compact and $\dim_{\mathbb{C}} X = 2$.

Proof. Twistor theory and deformations of holomorphic Poisson structures.

A **hyperkähler structure** is a Riemannian metric g with three complex structures I, J, K that are Kähler with respect to g and satisfy $IJK = -1$.
Idea. *Encode it with purely holomorphic data on an auxiliary space.*

- $x \in S^2 \subseteq \mathbb{R}^3 \implies I_x := x_1 I + x_2 J + x_3 K$ complex structure
- $Z = M \times S^2$ with $\mathbf{I}_{(q,x)} = (I_x, I_{S^2})$ is a complex manifold
- $p : Z \to S^2 = \mathbb{CP}^1$ holomorphic submersion
- I_x is Kähler with respect to g with Kähler form $\omega_x := x_1\omega_I + x_2\omega_J + x_3\omega_K$.
- $x, y, z \in S^2$ orthonormal basis $\implies (I_x, \omega_y + i\omega_z)$ holomorphic symplectic. Identifying $x \in S^2 \setminus \{N\}$ with $\zeta \in \mathbb{C} \subseteq \mathbb{CP}^1$, $\omega_y + i\omega_z$ is a multiple of
$$\Omega_\zeta := (\omega_J + i\omega_K) + 2i\zeta\omega_I + \zeta^2(\omega_J - i\omega_K).$$
Then, (I_ζ, Ω_ζ) are holomorphic symplectic structures for all $\zeta \in \mathbb{C}$. Ω_ζ extends to a global holomorphic section Ω of $\Lambda^2(\ker dp)^* \otimes p^*\mathcal{O}(2)$.
- $\tau : Z \to Z : (q, x) \mapsto (q, -x)$ is a real structure.
- Points $q \in M$ are identified with holomorphic sections $x \mapsto (q, x)$ of p fixed by τ with normal bundle $N \cong \mathcal{O}(1)^{\oplus 2n}$, called **real twistor lines**.

Definition

A **hyperkähler twistor space** is a complex manifold Z of dimension $2n+1$ together with

- a surjective holomorphic submersion $p : Z \to \mathbb{CP}^1$,
- a global holomorphic section Ω of $\Lambda^2(\ker dp)^* \otimes p^*\mathcal{O}(2)$ which restricts to a holomorphic symplectic form on each fibre $Z_\zeta = p^{-1}(\zeta)$.
- a real structure $\tau : Z \to Z$ covering the antipodal map $\mathbb{CP}^1 \to \mathbb{CP}^1$ and such that $\tau^*\bar{\Omega} = -\Omega$.

A **real twistor line** is a holomorphic section of p which is fixed by τ and whose normal bundle is isomorphic to $\mathcal{O}(1)^{\oplus 2n}$.

Theorem (Hitchin–Karlhede–Lindström–Roček 1987)

The space $\mathcal{M}$ of real twistor lines of a hyperkähler twistor space is a $4n$-dimensional hyperkähler manifold. Moreover, the evaluation maps

$$\mathcal{M} \longrightarrow Z_\zeta, \quad s \longmapsto s(\zeta)$$

are local diffeomorphisms.

Symplectic realization $M \to X$, $\mathrm{id} : X \hookrightarrow M$ complex Lagrangian.
Goal. Construct a hyperkähler structure on a neighbourhood of X in M.

Characterization of hyperkähler structures near complex Lagrangians:

Theorem (M.)

Let (M, I_0, Ω_0) be a holomorphic symplectic manifold and $X \subseteq M$ a complex Lagrangian submanifold. Suppose that there is a deformation (I_ζ, Ω_ζ) of holomorphic symplectic structures, for small $\zeta \in \mathbb{C}$, such that $\Omega_\zeta|_X = \zeta\omega$ for some Kähler form ω on X. Then, there is a hyperkähler structure (g, I, J, K) on a neighbourhood of X in M such that $\omega_I|_X = \omega$, $I = I_0$, and $\omega_J + i\omega_K = \Omega_0$.

Every hyperkähler structure can be obtained in this way using

$$\Omega_\zeta = (\omega_J + i\omega_K) + 2i\zeta\omega_I + \zeta^2(\omega_J - i\omega_K) = \Omega_0 + 2i\zeta\omega_I + \zeta^2\bar{\Omega}_0.$$

Proof sketch. $X \times S^1 \subseteq M \times \mathbb{C}$ is totally real ($\mathbb{R}^n \subseteq \mathbb{C}^n$).
$X \times S^1 \to \overline{X \times S^1} : (x, \zeta) \mapsto (x, -\zeta)$ extends to holomorphic map $M \times \mathbb{C}^* \to \overline{M \times \mathbb{C}^*}$. Glue $M \times \mathbb{C}$ and $\overline{M \times \mathbb{C}}$ to a twistor space $Z \to \mathbb{CP}^1$. For all $x \in X$, $\zeta \mapsto (x, \zeta)$ is a real twistor line. □

Summary.

- (M, I_0, Ω_0) holomorphic symplectic
- $X \subseteq M$ complex Lagrangian.
- Suppose $\exists$ deformation (I_ζ, Ω_ζ) such that $\Omega_\zeta|_X = \zeta\omega$, ω Kähler.

Twistor theory: There are diffeomorphisms

$$\varphi_\zeta : M \xrightarrow{\mathrm{ev}_0^{-1}} \{\text{real twistor lines}\} \xrightarrow{\mathrm{ev}_\zeta} M, \qquad (\zeta \in \mathbb{C})$$

such that $\varphi_\zeta^*\Omega_\zeta = \Omega_0 + 2i\zeta\omega_{I_0} + \zeta^2\bar{\Omega}_0$ and $g = \omega_{I_0}(\cdot, I_0\cdot)$ is hyperkähler.

- (X, I) compact Kähler manifold
- Deformations of I are obtained by solving the Maurer–Cartan equation

$$\bar{\partial}\phi + \frac{1}{2}[\phi, \phi] = 0, \qquad \phi(\zeta) = \sum_{n=1}^{\infty} \phi_n \zeta^n \in \Omega_X^{0,1}(T^{1,0}), \quad (\zeta \in \mathbb{C} \text{ small}).$$

$T_\zeta^{0,1} := (1 + \phi(\zeta))(T^{0,1})$ is the $(0,1)$-part of a new complex structure I_ζ.

- $[\phi_1] \in H^1(X, T^{1,0})$ determines the deformation up to diffeomorphisms.
- In general, some classes in $H^1(X, T^{1,0})$ can be obstructed. But:

Theorem (Hitchin 2012)

If π is a holomorphic Poisson structure on X and $\omega_1 \in \Omega_X^{1,1}$ is closed, then $[\pi\omega_1] \in H^1(X, T^{1,0})$ integrates to a deformation of complex structures I_ζ. Moreover, π is deformed to a holomorphic Poisson structure π_ζ on (X, I_ζ).

Proof. $\Omega_X^{1,1} \xrightarrow{\pi} \Omega_X^{0,1}(T^{1,0})$, $\omega \mapsto \pi\omega = \pi^{ij}\omega_{jk} d\bar{z}^k \otimes \frac{\partial}{\partial z^i} =: \phi$.

$$d\omega + \frac{1}{2}\partial i_\pi(\omega \wedge \omega) = 0 \qquad \Longrightarrow \qquad \bar{\partial}\phi + \frac{1}{2}[\phi, \phi] = 0.$$

For any closed $\omega_1 \in \Omega_X^{1,1}$ there exists $\omega_2, \omega_3, \omega_4, \ldots \in \Omega_X^{1,1}$ such that $\omega_\zeta := \sum_{n=1}^{\infty} \zeta^n \omega_n$ converges to a family of solutions.

A holomorphic Poisson structure (I, π) is determined by its **Dirac structure**:

$$L_{I,\pi} := \{v + \pi(\xi) + \xi : v \in T^{0,1}, \xi \in T^*_{1,0}\} \subseteq T_{\mathbb{C}}X \oplus T^*_{\mathbb{C}}X.$$

Encodes both I and π.

Gauge transformations. Given a Dirac structure L and a closed 2-form $\beta \in \Omega^2_X$,

$$e^{\beta}L := \{X + \xi + \beta(X) : (X, \xi) \in L\}$$

is a new Dirac structure.

Reinterpretation of Hitchin's theorem (Gualtieri 2018). *For any closed $\omega_1 \in \Omega^{1,1}_X$ there is a family of gauge transformations*

$$\beta_\zeta = \zeta\omega_1 + \zeta^2\beta_2 + \zeta^3\beta_3 + \cdots \in \Omega^{2,closed}_X, \quad (\zeta \in \mathbb{C} \textit{ small})$$

such that $e^{\beta_\zeta}L_{I,\pi} = L_{I_\zeta,\pi_\zeta}$ for new holomorphic Poisson structures (I_ζ, π_ζ).

If $\omega_\zeta \in \Omega^{1,1}_X$ are solutions to $d\omega_\zeta + \frac{1}{2}\partial i_\pi(\omega_\zeta \wedge \omega_\zeta) = 0$, then

$$\beta_\zeta = \omega_\zeta + \frac{1}{2} i_\pi(\omega_\zeta \wedge \omega_\zeta).$$

Hyperkähler structures on symplectic realizations

- $(M, \Omega_0) \to (X, \pi)$ symplectic realization, $\mathrm{id} : X \hookrightarrow M$ complex Lagrangian
- **Goal.** Find a deformation $(M, I_\zeta, \Omega_\zeta)$ such that $\mathrm{id}^*\Omega_\zeta = \zeta\omega_1$, ω_1 Kähler
- Take any Kähler form ω_1 on X and consider the **Hitchin deformation in the direction of ω_1:** $[\pi\omega_1] \in H^1(X, T^{1,0})$ tangent to a deformation (X, I_ζ, π_ζ), determined by gauge transformations $\beta_\zeta \in \Omega^2_X$.
- **Lift the deformation from X to M:** For small $\zeta_1, \zeta_2 \in \mathbb{C}$,
$$\Omega_{\zeta_1,\zeta_2} := \Omega_0 + s^*\beta_{\zeta_1} - t^*\beta_{\zeta_2}$$
is a holomorphic symplectic form for a unique complex structure I_{ζ_1,ζ_2}.
Remark. $(M, \Omega_{\zeta_1,\zeta_2})$ is a dual pair between (X, π_{ζ_1}) and (X, π_{ζ_2})
- $\mathrm{id}^*\Omega_{\zeta_1,\zeta_2} = \mathrm{id}^*\Omega_0 + \mathrm{id}^* s^*\beta_{\zeta_1} - \mathrm{id}^* t^*\beta_{\zeta_2} = \beta_{\zeta_1} - \beta_{\zeta_2}$
- $\mathrm{id}^*\Omega_{\zeta,-\zeta} = \beta_\zeta - \beta_{-\zeta} = 2(\zeta\omega_1 + \zeta^3\beta_3 + \zeta^5\beta_5 + \zeta^7\beta_7 + \cdots)$
- **Conclusion.** We need $\beta_3 = \beta_5 = \beta_7 = \beta_9 = \cdots = 0.$
- Can be done when $\dim_{\mathbb{C}} X = 2$ (starting with any ω_1).

Summary. Let (X, π) be a compact holomorphic Poisson manifold together with a real-analytic Kähler form ω_1.
Hitchin's unobstructedness theorem: There is a family $\omega_\zeta = \zeta\omega_1 + \zeta^2\omega_2 + \zeta^3\omega_3 + \cdots \in \Omega_X^{1,1}$ solving

$$d\omega_\zeta + \tfrac{1}{2}\partial i_\pi(\omega_\zeta \wedge \omega_\zeta) = 0. \tag{1}$$

Then, $\beta_\zeta := \omega_\zeta + \frac{1}{2} i_\pi(\omega_\zeta \wedge \omega_\zeta) \in \Omega_X^{2,\text{closed}}$ are gauge transformations defining new holomorphic Poisson structures $e^{\beta_\zeta} L_{I,\pi} = L_{I_\zeta,\pi_\zeta}$.

Proposition

If β_ζ has no odd degree term other than $\zeta\omega_1$, i.e.

$$i_\pi(\omega_{2n} \wedge \omega_1) = 0 \quad \text{and} \quad \omega_{2n+1} = 0 \quad \text{for all } n \geq 1, \tag{2}$$

then any symplectic realization $(M, \Omega_0) \to (X, \pi)$ has a hyperkähler structure (g, I, J, K) on a neighbourhood of X s.t. $\omega_I|_X = \omega_1$ and $\Omega_0 = \omega_J + i\omega_K$.

Example (Feix–Kaledin hyperkähler structures)

If $\pi = 0$ then $\omega_\zeta := \zeta\omega_1$ is a solution to (1) and (2). So T^*X has a hyperkähler structure on a neighbourhood of its zero section.

- (X, I, π) compact holomorphic Poisson manifold
- ω_1 real-analytic Kähler form
- $\Delta = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$ Laplacian of ω_1; G Green operator of Δ

Solutions $\omega_\zeta = \sum_{n=1}^{\infty} \zeta^n \omega_n$ to

$$d\omega_\zeta + \frac{1}{2} \partial i_\pi (\omega_\zeta \wedge \omega_\zeta) = 0$$

can be obtained by defining $\omega_2, \omega_3, \omega_4, \ldots$ recursively by

$$\omega_n = \frac{1}{2} \partial \bar{\partial}^* G \sum_{i+j=n} i_\pi (\omega_i \wedge \omega_j).$$

We need $i_\pi(\omega_{2n} \wedge \omega_1) = 0$ and $\omega_{2n+1} = 0$ for all $n \geq 1$.
If $\dim_{\mathbb{C}} X = 2$, then the Kähler identities give

$$\omega_n \wedge \omega_1 = L(\omega_n) = \frac{1}{2} \partial \bar{\partial}^* L G \sum_{i+j=n} i_\pi (\omega_i \wedge \omega_j) = 0,$$

since $G \sum_{i+j=n} i_\pi(\omega_i \wedge \omega_j) \in \Omega_X^{0,2}$ and $L(\Omega_X^{0,2}) \subseteq \Omega_X^{1,3} = 0$.
Then, by induction, $\omega_{2n+1} = 0$ for all $n \geq 1$.

Theorem (M.)

Let (X, π) be a compact holomorphic Poisson surface endowed with a real-analytic Kähler form ω.

Let $(M, \Omega) \to (X, \pi)$ be a holomorphic symplectic realization with Lagrangian section $\mathrm{id} : X \hookrightarrow M$.

Then, there is a hyperkähler structure (g, I, J, K) on a neighbourhood of X in M such that $\mathrm{id}^*\omega_I = \omega$ and $\Omega = \omega_J + i\omega_K$.

For each $\zeta \in \mathbb{C}$, the holomorphic symplectic structure $(M, I_\zeta, \Omega_\zeta)$ is a dual pair between the Hitchin deformations π_ζ and $\pi_{-\zeta}$ in the direction of ω:

$$\begin{array}{ccc} & (M, I_\zeta, \Omega_\zeta) & \\ {\scriptstyle s} \swarrow & & \searrow {\scriptstyle t} \\ (X, \pi_\zeta) & & (X, \pi_{-\zeta}). \end{array}$$

thank you