Deformations of Lagrangian submanifolds in log-symplectic manifolds

Stephane Geudens
joint work with Marco Zambon

KU Leuven

arXiv:2009.01146

Lagrangian submanifolds in symplectic geometry

Weinstein's Lagrangian neighborhood theorem

Around a Lagrangian L,

$$(M, \omega) \cong (T^*L, \omega_{can}).$$

Lagrangian submanifolds in symplectic geometry

Weinstein's Lagrangian neighborhood theorem

Around a Lagrangian L,

$$(M, \omega) \cong (T^*L, \omega_{can}).$$

In the local model (T^*L, ω_{can}):

- $\mathrm{Gr}(\alpha) \subset (T^*L, \omega_{can})$ is Lagrangian iff. $d\alpha = 0$.
- $\mathrm{Gr}(\alpha), \mathrm{Gr}(\beta) \subset (T^*L, \omega_{can})$ related by Hamiltonian diffeomorphism iff. $[\alpha] = [\beta]$ in $H^1(L)$.

$$\mathcal{M}^{Ham}(L) = H^1(L).$$

Lagrangian submanifolds in symplectic geometry

Weinstein's Lagrangian neighborhood theorem

Around a Lagrangian L,

$$(M, \omega) \cong (T^*L, \omega_{can}).$$

In the local model (T^*L, ω_{can}):

- $\mathrm{Gr}(\alpha) \subset (T^*L, \omega_{can})$ is Lagrangian iff. $d\alpha = 0$.
- $\mathrm{Gr}(\alpha), \mathrm{Gr}(\beta) \subset (T^*L, \omega_{can})$ related by Hamiltonian diffeomorphism iff. $[\alpha] = [\beta]$ in $H^1(L)$.

$$\mathcal{M}^{Ham}(L) = H^1(L).$$

What about the log-symplectic case?

Lagrangian submanifolds in Poisson geometry

Definition[1]
$L \subset (M, \pi)$ is Lagrangian if for all $p \in L$:

$$T_pL \cap T_pS \text{ is a Lagrangian subspace of } (T_pS, (\omega_S)_p).$$

Here (S, ω_S) is the symplectic leaf through $p \in L$.

[1] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progress in Mathematics, Birkhäuser, 1994

Lagrangian submanifolds in Poisson geometry

Definition[1]

$L \subset (M, \pi)$ is Lagrangian if for all $p \in L$:

$$T_pL \cap T_pS \text{ is a Lagrangian subspace of } (T_pS, (\omega_S)_p).$$

Here (S, ω_S) is the symplectic leaf through $p \in L$.

Examples

- Planes through the origin in $(\mathfrak{so}_3^*, \pi_{lin})$.
- Graphs of Poisson immersions $\phi : (M_1, \pi_1) \to (M_2, \pi_2)$.

[1] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progress in Mathematics, Birkhäuser, 1994

Log-symplectic manifolds

Definition

(M^{2n}, π) is log-symplectic if $\wedge^n \pi : M \to \wedge^{2n} TM$ is transverse to the zero section.

π is symplectic away from singular locus $Z := (\wedge^n \pi)^{-1}(0)$.

- Z is a hypersurface with induced corank-one Poisson structure.
- $(Z, \pi|_Z)$ has a Poisson vector field transverse to the symplectic leaves: $V_{mod}|_Z$.

Log-symplectic manifolds

Definition

(M^{2n}, π) is log-symplectic if $\wedge^n \pi : M \to \wedge^{2n} TM$ is transverse to the zero section.

π is symplectic away from singular locus $Z := (\wedge^n \pi)^{-1}(0)$.

- Z is a hypersurface with induced corank-one Poisson structure.
- $(Z, \pi|_Z)$ has a Poisson vector field transverse to the symplectic leaves: $V_{mod}|_Z$.

Example

On $(\mathbb{R}^{2n}, x_1, y_1, \dots, x_n, y_n)$:

$$\pi = \partial_{x_1} \wedge y_1 \partial_{y_1} + \sum_{i=2}^{n} \partial_{x_i} \wedge \partial_{y_i}.$$

Modular vector field is ∂_{x_1}. This is the local model around $p \in Z$.

Lagrangian submanifolds in log-symplectic geometry

$L \subset (M^{2n}, Z, \pi)$ Lagrangian.

[2] C. Kirchhoff-Lukat, *Aspects of Generalized Geometry: Branes with Boundary, Blow-ups, Brackets and Bundles*, PhD thesis, University of Cambridge, 2018.

Lagrangian submanifolds in log-symplectic geometry

$L \subset (M^{2n}, Z, \pi)$ Lagrangian.

- If $L \pitchfork Z$: use b-symplectic geometry[2]. Around L:

$$(M, \omega) \cong ({}^bT^*L, \omega_{can}).$$

Hence $\mathcal{M}^{Ham}(L) = {}^bH^1(L) \cong H^1(L) \oplus H^0(L \cap Z)$.

[2] C. Kirchhoff-Lukat, *Aspects of Generalized Geometry: Branes with Boundary, Blow-ups, Brackets and Bundles*, PhD thesis, University of Cambridge, 2018.

Lagrangian submanifolds in log-symplectic geometry

$L \subset (M^{2n}, Z, \pi)$ Lagrangian.

- If $L \pitchfork Z$: use b-symplectic geometry[2]. Around L:

  $$(M, \omega) \cong ({}^b T^* L, \omega_{can}).$$

  Hence $\mathcal{M}^{Ham}(L) = {}^b H^1(L) \cong H^1(L) \oplus H^0(L \cap Z)$.

- If $L \subset Z$ then
  - either $\dim L = n - 1$ and components of L lie inside leaves,
  - or $\dim L = n$ and L is transverse to the leaves of Z.

[2] C. Kirchhoff-Lukat, *Aspects of Generalized Geometry: Branes with Boundary, Blow-ups, Brackets and Bundles*, PhD thesis, University of Cambridge, 2018.

Lagrangian submanifolds in log-symplectic geometry

$L \subset (M^{2n}, Z, \pi)$ Lagrangian.

- If $L \pitchfork Z$: use b-symplectic geometry[2]. Around L:

$$(M, \omega) \cong ({}^b T^* L, \omega_{can}).$$

  Hence $\mathcal{M}^{Ham}(L) = {}^b H^1(L) \cong H^1(L) \oplus H^0(L \cap Z)$.

- If $L \subset Z$ then
  - either $\dim L = n - 1$ and components of L lie inside leaves,
  - or $\dim L = n$ and L is transverse to the leaves of Z.

We focus on $L^n \subset Z \subset M^{2n}$.
L^n compact, connected.

[2] C. Kirchhoff-Lukat, *Aspects of Generalized Geometry: Branes with Boundary, Blow-ups, Brackets and Bundles*, PhD thesis, University of Cambridge, 2018.

Normal form for log-symplectic structure around L

Construct normal form in two steps: $L \subset (Z, \pi|_Z)$ and $Z \subset (M, \pi)$.

Normal form for log-symplectic structure around L

Construct normal form in two steps: $L \subset (Z, \pi|_Z)$ and $Z \subset (M, \pi)$.

Step 1: General fact

$L \subset (N, \pi)$ Lagrangian transverse to leaves. Then

Normal form for log-symplectic structure around L

Construct normal form in two steps: $L \subset (Z, \pi|_Z)$ and $Z \subset (M, \pi)$.

Step 1: General fact

$L \subset (N, \pi)$ Lagrangian transverse to leaves. Then

- there is a neighborhood U of L s.t. $\pi|_U$ is regular.
  $\to L$ gets foliation $\mathcal{F}_L$.

Normal form for log-symplectic structure around L

Construct normal form in two steps: $L \subset (Z, \pi|_Z)$ and $Z \subset (M, \pi)$.

Step 1: General fact

$L \subset (N, \pi)$ Lagrangian transverse to leaves. Then

- there is a neighborhood U of L s.t. $\pi|_U$ is regular.
  $\to L$ gets foliation $\mathcal{F}_L$.
- there is a canonical Poisson structure π_{can} on $T^*\mathcal{F}_L$ s.t.

$$T^*\mathcal{F}_L = \coprod_{B \in \mathcal{F}_L} (T^*B, \omega_{T^*B}).$$

Normal form for log-symplectic structure around L

Construct normal form in two steps: $L \subset (Z, \pi|_Z)$ and $Z \subset (M, \pi)$.

Step 1: General fact

$L \subset (N, \pi)$ Lagrangian transverse to leaves. Then

- there is a neighborhood U of L s.t. $\pi|_U$ is regular.
  $\rightarrow L$ gets foliation $\mathcal{F}_L$.
- there is a canonical Poisson structure π_{can} on $T^*\mathcal{F}_L$ s.t.

$$T^*\mathcal{F}_L = \coprod_{B \in \mathcal{F}_L} (T^*B, \omega_{T^*B}).$$

- around L,

$$(N, \pi) \cong (T^*\mathcal{F}_L, \pi_{can}).$$

Normal form for log-symplectic structure around L

Step 2: $Z \subset (M, \pi)$[3]

Let (M, Z, π) be an orientable log-symplectic manifold. The local model for (M, π) around Z is $Z \times \mathbb{R}$ with

$$V_{mod}|_Z \wedge t\partial_t + \pi|_Z.$$

[3] V. Guillemin, E. Miranda, A.R. Pires, *Symplectic and Poisson geometry on b-manifolds*, Adv. Math. **264**, p. 864-896 (2014).

Normal form for log-symplectic structure around L

Step 2: $Z \subset (M, \pi)$[3]

Let (M, Z, π) be an orientable log-symplectic manifold. The local model for (M, π) around Z is $Z \times \mathbb{R}$ with

$$V_{mod}|_Z \wedge t\partial_t + \pi|_Z.$$

Corollary (Normal form ad interim)

The local model around $L^n \subset Z \subset (M^{2n}, \pi)$ is $T^*\mathcal{F}_L \times \mathbb{R}$ with

$$V \wedge t\partial_t + \pi_{can}.$$

Here V is image of $V_{mod}|_Z$ under $(Z, \pi|_Z) \xrightarrow{\sim} (T^*\mathcal{F}_L, \pi_{can})$.

We can choose any representative of $[V] \in H^1_{\pi_{can}}(T^*\mathcal{F}_L)$...

[3] V. Guillemin, E. Miranda, A.R. Pires, *Symplectic and Poisson geometry on b-manifolds*, Adv. Math. **264**, p. 864-896 (2014).

Intermezzo: Poisson vector fields on $(T^*\mathcal{F}_L, \pi_{can})$

$(L, \mathcal{F}_L)$ foliated manifold. Denote

$$\mathfrak{X}(L)^{\mathcal{F}_L} := \{W \in \mathfrak{X}(L) : [W, \Gamma(T\mathcal{F}_L)] \subset \Gamma(T\mathcal{F}_L)\}.$$

The cotangent lift of $W \in \mathfrak{X}(L)^{\mathcal{F}_L}$ pushes forward under $T^*L \to T^*\mathcal{F}_L$ to a Poisson vector field $\widetilde{W}$ on $(T^*\mathcal{F}_L, \pi_{can})$.

Intermezzo: Poisson vector fields on $(T^*\mathcal{F}_L, \pi_{can})$

$(L, \mathcal{F}_L)$ foliated manifold. Denote

$$\mathfrak{X}(L)^{\mathcal{F}_L} := \{W \in \mathfrak{X}(L) : [W, \Gamma(T\mathcal{F}_L)] \subset \Gamma(T\mathcal{F}_L)\}.$$

The cotangent lift of $W \in \mathfrak{X}(L)^{\mathcal{F}_L}$ pushes forward under $T^*L \to T^*\mathcal{F}_L$ to a Poisson vector field $\widetilde{W}$ on $(T^*\mathcal{F}_L, \pi_{can})$.

Proposition

The first Poisson cohomology of $(T^*\mathcal{F}_L, \pi_{can})$ is:

$$\begin{array}{ccc} H^1_{\pi_{can}}(T^*\mathcal{F}_L) \cong \mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L) \times H^1(\mathcal{F}_L) : \\ [\widetilde{X} + \pi_{can}^\sharp(p^*\gamma)] \quad \longleftarrow \qquad ([X], [\gamma]). \end{array}$$

Normal form for log-symplectic structure around L

Assume that $[V] \longleftrightarrow ([X], [\gamma])$ under

$$H^1_{\pi_{can}}(T^*\mathcal{F}_L) \cong \mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L) \times H^1(\mathcal{F}_L).$$

Normal form for log-symplectic structure around L

Assume that $[V] \longleftrightarrow ([X],[\gamma])$ under

$$H^1_{\pi_{can}}(T^*\mathcal{F}_L) \cong \mathfrak{X}(L)^{\mathcal{F}_L}/\Gamma(T\mathcal{F}_L) \times H^1(\mathcal{F}_L).$$

Corollary (Normal form)

The local model around $L^n \subset Z \subset (M^{2n},\pi)$ is $T^*\mathcal{F}_L \times \mathbb{R}$ with log-symplectic structure

$$(\widetilde{X} + \pi^\sharp_{can}(p^*\gamma)) \wedge t\partial_t + \pi_{can}.$$

Lagrangian deformations

Look at Lagrangian sections $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ in the local model

$$\Big(T^*\mathcal{F}_L \times \mathbb{R},\ (\widetilde{X} + \pi^\sharp_{can}(p^*\gamma)) \wedge t\partial_t + \pi_{can} \Big).$$

Lagrangian deformations

Look at Lagrangian sections $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ in the local model

$$\Big(T^*\mathcal{F}_L \times \mathbb{R},\ (\widetilde{X} + \pi^\sharp_{can}(p^*\gamma)) \wedge t\partial_t + \pi_{can}\Big).$$

Proposition

The image of a section $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ is Lagrangian exactly when

$$\begin{cases} d_{\mathcal{F}_L}\alpha = 0 \\ d_{\mathcal{F}_L} f + f(\gamma - \mathcal{L}_X\alpha) = 0 \end{cases} .$$

Remarks

- If $\eta \in \Omega^1(\mathcal{F}_L)$ is closed, then we get a differential

$$d^{\eta}_{\mathcal{F}_L} \bullet := d_{\mathcal{F}_L} \bullet + \eta \wedge \bullet.$$

  Denote the cohomology by $H^{\bullet}_{\eta}(\mathcal{F}_L)$.

Remarks

- If $\eta \in \Omega^1(\mathcal{F}_L)$ is closed, then we get a differential

$$d^{\eta}_{\mathcal{F}_L} \bullet := d_{\mathcal{F}_L} \bullet + \eta \wedge \bullet.$$

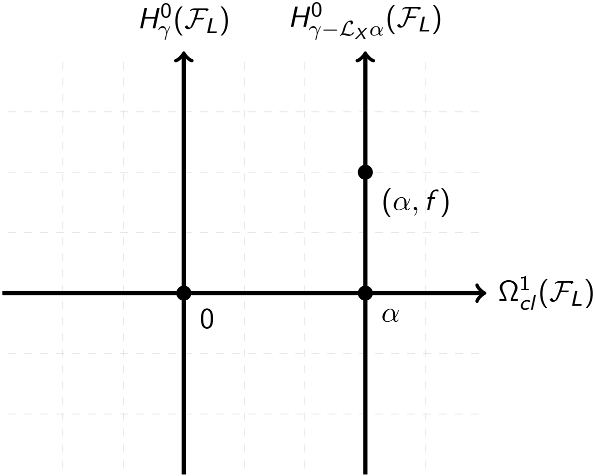
  Denote the cohomology by $H^{\bullet}_{\eta}(\mathcal{F}_L)$.

- So $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ is Lagrangian exactly when

$$\begin{cases} d_{\mathcal{F}_L}\alpha = 0 \\ d^{\gamma - \mathcal{L}_X \alpha}_{\mathcal{F}_L} f = 0 \end{cases}.$$

Deforming L into $Graph(\alpha, f)$ can be done in two steps:

1. Deform L inside singular locus along $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$.
2. Push $Graph(\alpha)$ out of singular locus along $f \in H^0_{\gamma - \mathcal{L}_X \alpha}(\mathcal{F}_L)$.



The DGLA behind the deformation problem

The equations for Lagrangian sections $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ are the Maurer-Cartan equation of a DGLA.

The DGLA behind the deformation problem

The equations for Lagrangian sections $(\alpha, f) \in \Gamma(T^*\mathcal{F}_L \times \mathbb{R})$ are the Maurer-Cartan equation of a DGLA.

Proposition

The deformation problem of the Lagrangian L is governed by a DGLA structure on $\Gamma\big(\wedge^\bullet(T^*\mathcal{F}_L \times \mathbb{R})\big) = \Gamma(\wedge^\bullet T^*\mathcal{F}_L \oplus \wedge^{\bullet-1} T^*\mathcal{F}_L)$ whose structure maps $(d, [\![\cdot,\cdot]\!])$ are defined by

$$
\begin{aligned}
d : \Gamma\left(\wedge^k \left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) &\to \Gamma\left(\wedge^{k+1}\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) : \\
&(\alpha, \beta) \mapsto \left(-d_{\mathcal{F}_L}\alpha, -d_{\mathcal{F}_L}\beta - \gamma \wedge \beta\right), \\
[\![\cdot,\cdot]\!] : \Gamma\left(\wedge^k \left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) \otimes \Gamma\left(\wedge^l \left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) &\to \Gamma\left(\wedge^{k+l}\left(T^*\mathcal{F}_L \times \mathbb{R}\right)\right) : \\
&(\alpha, \beta) \otimes (\delta, \epsilon) \mapsto \left(0, \mathcal{L}_X\alpha \wedge \epsilon - (-1)^{kl}\mathcal{L}_X\delta \wedge \beta\right).
\end{aligned}
$$

So $Gr(\alpha, f)$ is Lagrangian iff. $d(\alpha, f) + \frac{1}{2}[\![(\alpha, f), (\alpha, f)]\!] = 0.$

Some geometric aspects of the deformation problem

Some geometric aspects of the deformation problem

1. When do small deformations stay inside the singular locus?

Some geometric aspects of the deformation problem

1. When do small deformations stay inside the singular locus?
2. Is the deformation problem obstructed?

Some geometric aspects of the deformation problem

1. When do small deformations stay inside the singular locus?
2. Is the deformation problem obstructed?
(3. Moduli spaces.)

Some geometric aspects of the deformation problem

1. When do small deformations stay inside the singular locus?
2. Is the deformation problem obstructed?

(3. Moduli spaces.)

Recall the two options:

1. $(L, \mathcal{F}_L)$ is the foliation of a fibration $L \to S^1$.
2. all leaves of $(L, \mathcal{F}_L)$ are dense.

Deformations constrained to the singular locus

Require that $H^0_{\gamma - \mathcal{L}_X \alpha}(\mathcal{F}_L) = 0$ for small $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$.

Deformations constrained to the singular locus

Require that $H^0_{\gamma - \mathcal{L}_X \alpha}(\mathcal{F}_L) = 0$ for small $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$.

Lemma

Let $\eta \in \Omega^1(\mathcal{F}_L)$ be leafwise closed.

Deformations constrained to the singular locus

Require that $H^0_{\gamma-\mathcal{L}_X\alpha}(\mathcal{F}_L) = 0$ for small $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$.

Lemma

Let $\eta \in \Omega^1(\mathcal{F}_L)$ be leafwise closed.

- If $\mathcal{F}_L$ is given by fibration $p : L \to S^1$ then $H^1(\mathcal{F}_L) \cong \Gamma(\mathcal{H}^1)$, where
$$\mathcal{H}^1_q = H^1(p^{-1}(q)).$$
Then
$$H^0_\eta(\mathcal{F}_L) \cong \{f \in C^\infty(S^1) : f \cdot [\eta] = 0\}.$$

Deformations constrained to the singular locus

Require that $H^0_{\gamma-\mathcal{L}_X\alpha}(\mathcal{F}_L) = 0$ for small $\alpha \in \Omega^1_{cl}(\mathcal{F}_L)$.

Lemma

Let $\eta \in \Omega^1(\mathcal{F}_L)$ be leafwise closed.

- If $\mathcal{F}_L$ is given by fibration $p : L \to S^1$ then $H^1(\mathcal{F}_L) \cong \Gamma(\mathcal{H}^1)$, where
$$\mathcal{H}^1_q = H^1(p^{-1}(q)).$$
Then
$$H^0_\eta(\mathcal{F}_L) \cong \{f \in C^\infty(S^1) : f \cdot [\eta] = 0\}.$$

- If the leaves of $\mathcal{F}_L$ are dense, then
$$H^0_\eta(\mathcal{F}_L) = \begin{cases} \mathbb{R} & \text{if } \eta \text{ is exact} \\ 0 & \text{otherwise} \end{cases}.$$

Proposition

If $[\gamma] = 0$ in $H^1(\mathcal{F}_L)$, there is a path L_s not contained in the singular locus for $s > 0$.

Proposition

If $[\gamma] = 0$ in $H^1(\mathcal{F}_L)$, there is a path L_s not contained in the singular locus for $s > 0$.

Proposition

- Suppose $\mathcal{F}_L$ is given by a fibration $L \to S^1$. If for each leaf B of $\mathcal{F}_L$, $[\gamma|_B] \neq 0 \in H^1(B)$, then $\mathcal{C}^1$-small deformations of L stay inside the singular locus.

Proposition

If $[\gamma] = 0$ in $H^1(\mathcal{F}_L)$, there is a path L_s not contained in the singular locus for $s > 0$.

Proposition

- Suppose $\mathcal{F}_L$ is given by a fibration $L \to S^1$. If for each leaf B of $\mathcal{F}_L$, $[\gamma|_B] \neq 0 \in H^1(B)$, then $\mathcal{C}^1$-small deformations of L stay inside the singular locus.
- Suppose $\mathcal{F}_L$ has dense leaves, and that $H^1(\mathcal{F}_L)$ is finite dimensional. If γ is not exact, then $\mathcal{C}^\infty$-small deformations of L stay inside the singular locus.

Example

Consider $(\mathbb{T}^2 \times \mathbb{R}^2, \theta_1, \theta_2, x_1, x_2)$ with log-symplectic structure

$$\pi = (\partial_{\theta_1} + \partial_{x_2}) \wedge x_1 \partial_{x_1} + \partial_{\theta_2} \wedge \partial_{x_2}$$

and $L := \mathbb{T}^2$. The leaves of $\mathcal{F}_L$ are fibers of $(\mathbb{T}^2, \theta_1, \theta_2) \to (S^1, \theta_1)$. As $\gamma = d\theta_2$, small deformations of $\mathbb{T}^2$ stay inside the singular locus.

Example

Consider $(\mathbb{T}^2 \times \mathbb{R}^2, \theta_1, \theta_2, x_1, x_2)$ with log-symplectic structure

$$\pi = (\partial_{\theta_1} + \partial_{x_2}) \wedge x_1 \partial_{x_1} + \partial_{\theta_2} \wedge \partial_{x_2}$$

and $L := \mathbb{T}^2$. The leaves of $\mathcal{F}_L$ are fibers of $(\mathbb{T}^2, \theta_1, \theta_2) \to (S^1, \theta_1)$. As $\gamma = d\theta_2$, small deformations of $\mathbb{T}^2$ stay inside the singular locus.

Non-example

Consider $(\mathbb{T}^2 \times \mathbb{R}^2, \theta_1, \theta_2, x_1, x_2)$ with log-symplectic structure

$$\pi = (V + \partial_{\theta_1}) \wedge x_1 \partial_{x_1} + (\lambda \partial_{\theta_1} + \partial_{\theta_2}) \wedge \partial_{x_2}.$$

Here $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ is a Liouville number and V is a suitable Poisson vector field on the singular locus. Let $L := \mathbb{T}^2$, so $\mathcal{F}_L$ is the Kronecker foliation. For any $k \geq 0$, there are arbitrarily $\mathcal{C}^k$-small deformations of L not contained in the singular locus.

Obstructedness of first order deformations

A deformation problem governed by a DGLA $(W, d, [\![\cdot, \cdot]\!])$ is unobstructed if any closed $w \in W_1$ is tangent to a curve of Maurer-Cartan elements.

Obstructedness of first order deformations

A deformation problem governed by a DGLA $(W, d, [\![\cdot,\cdot]\!])$ is unobstructed if any closed $w \in W_1$ is tangent to a curve of Maurer-Cartan elements.

Definition

The Kuranishi map of $(W, d, [\![\cdot,\cdot]\!])$ is

$$Kr : H^1(W) \to H^2(W) : [w] \mapsto \big[[\![w, w]\!]\big].$$

Proposition

$$w \text{ unobstructed} \Rightarrow Kr[w] = 0.$$

An obstructed example

Consider $(\mathbb{T}^2 \times \mathbb{R}^2, \theta_1, \theta_2, x_1, x_2)$ with log-symplectic structure

$$\pi = \partial_{\theta_1} \wedge x_1 \partial_{x_1} + \partial_{\theta_2} \wedge \partial_{x_2}$$

and $L := \mathbb{T}^2 \times \{(0,0)\}$. Here $X = \partial_{\theta_1}$ and $\gamma = 0$.

The Kuranishi map reads

$$Kr[(gd\theta_2, f)] = \left[\left(0, 2f\frac{\partial g}{\partial \theta_1} d\theta_2\right)\right] \in H^2(\mathcal{F}_L) \oplus H^1(\mathcal{F}_L).$$

So

$$Kr[(gd\theta_2, f)] = 0 \Leftrightarrow \int_{S^1} f\frac{\partial g}{\partial \theta_1} d\theta_2 = 0.$$

<u>e.g.:</u> $(\sin(\theta_1)d\theta_2, \cos(\theta_1))$ is an obstructed first order deformation.

Criteria for unobstructedness

Proposition

For a first order deformation $(\alpha, f) \in \Omega^1(\mathcal{F}_L) \times C^\infty(L)$, the following are equivalent:

1. (α, f) is smoothly unobstructed,
2. $Kr[(\alpha, f)] = 0$,
3. $\mathcal{L}_X \alpha$ is exact on $L \setminus \mathcal{Z}_f$,
4. α extends to a closed one-form on $L \setminus \mathcal{Z}_f$.

Criteria for unobstructedness

Proposition

For a first order deformation $(\alpha, f) \in \Omega^1(\mathcal{F}_L) \times C^\infty(L)$, the following are equivalent:

1. (α, f) is smoothly unobstructed,
2. $Kr[(\alpha, f)] = 0$,
3. $\mathcal{L}_X \alpha$ is exact on $L \setminus \mathcal{Z}_f$,
4. α extends to a closed one-form on $L \setminus \mathcal{Z}_f$.

Example

Consider $(\mathbb{T}^2 \times \mathbb{R}^2, \theta_1, \theta_2, x_1, x_2)$ with log-symplectic structure

$$\pi = \partial_{\theta_1} \wedge x_1 \partial_{x_1} + (\lambda \partial_{\theta_1} + \partial_{\theta_2}) \wedge \partial_{x_2},$$

for $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. Then $L = \mathbb{T}^2$ is Lagrangian with $T\mathcal{F}_L = \ker(d\theta_1 - \lambda d\theta_2)$.

- For generic λ, the deformation problem is unobstructed.
- For Liouville λ, the deformation problem is obstructed.

Thanks!

P.S.: Definition

$\lambda \in \mathbb{R}$ is a Liouville number if for all integers $p \geq 1$, there exist $m_p, n_p \in \mathbb{Z}$ such that $n_p > 1$ and

$$0 < \left| \lambda - \frac{m_p}{n_p} \right| < \frac{1}{n_p^p}.$$