On two notions of a gerbe over a stack

Praphulla Koushik

School of Mathematics,
IISER Thiruvananthapuram, India

October 16, 2020

Joint work with Saikat Chatterjee

Aim

To investigate the correspondence between two notions of a gerbe over a (differentiable) stack, as appeared in the following papers:

- Kai Behrend, Ping Xu, *Differentiable stacks and gerbes*. Journal of Symplectic Geometry 9 (2011), no. 3, Page 285 – 341.
- Camille Laurent-Gengoux, Mathieu Stiénon, Ping Xu, *Non-abelian differentiable gerbes*, Advances in Mathematics, Volume 220, Issue 5, 2009, Pages 1357 – 1427.

Motivation and Plan

The above mentioned papers which declare different meanings for the phrase “gerbe over a (differetiable) stack” introduces some constructions over a gerbe over a stack, for example, a connection over a gerbe over a stack.

We believe that an explicit correspondence between these two notions of a gerbe over a stack would help to improve our understanding about those geometric structures (and other related constructions).

1. Lie groupoids.
2. Differentiable stacks.
3. correspondence between Lie groupoids and differentiable stacks.
4. morphism of Lie groupoids and morphisms of differentiable stacks.

Lie groupoids

Definition (Lie groupoid)

A *Lie groupoid* is a groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$, where $\mathcal{G}_0, \mathcal{G}_1$ are smooth manifolds such that the maps $s, t : \mathcal{G}_1 \to \mathcal{G}_0$ are submersions and other structure maps are smooth.

Definition (action of a Lie groupoid on a manifold)

Let $\mathcal{G}$ be a Lie groupoid and P be a manifold. A *right action of* $\mathcal{G}$ *on* P consists of a pair of smooth maps $(a_{\mathcal{G}} : P \to \mathcal{G}_0, \mu : P \times_{\mathcal{G}_0, t} \mathcal{G}_1 \to P)$ satisfying the usual conditions of group actions.

Example (Action of a Lie groupoid $[G \rightrightarrows *]$ on a manifold P)

Let G be a Lie group and $[G \rightrightarrows *]$ be the associated Lie groupoid. Let P be a manifold. Then, an action of $[G \rightrightarrows *]$ on P is precisely the action of the Lie group G on the manifold P.

Examples of action of a Lie groupoid on a manifold

Example (Action of a Lie groupoid $\mathcal{G}$ on $\mathcal{G}_1$)

Let $\mathcal{G}$ be a Lie groupoid, then the composition map $m : \mathcal{G}_1 \times_{s,\mathcal{G}_0,t} \mathcal{G}_1 \to \mathcal{G}_1$ defines a right action of $\mathcal{G}$ on $\mathcal{G}_1$ given by $\mathcal{G}_1 \times_{s,\mathcal{G}_0,t} \mathcal{G}_1 \to \mathcal{G}_1$, a left action of $\mathcal{G}$ on $\mathcal{G}_1$ given by $\mathcal{G}_1 \times_{s,\mathcal{G}_0,t} \mathcal{G}_1 \to \mathcal{G}_1$.

Example (Action of a Lie groupoid $[G \times M \rightrightarrows M]$ on a manifold P)

Let G be a Lie group acting on a manifold M. Let $[G \times M \rightrightarrows M]$ be the associated Lie groupoid. Let P be a manifold. An action of $[G \times M \rightrightarrows M]$ on the manifold P is precisely given by an action of G on P and a G-equivariant smooth map $\pi : P \to M$.

principal $\mathcal{G}$-bundle over a manifold

Definition (principal $\mathcal{G}$-bundle over a manifold)

Let $\mathcal{G}$ be a Lie groupoid and M be a manifold. A *principal $\mathcal{G}$-bundle over M* consists of,

1. a smooth manifold P,
2. a right action of $\mathcal{G}$ on P given by $(a_{\mathcal{G}} : P \to \mathcal{G}_0, \mu : P \times_{\mathcal{G}_0,t} \mathcal{G}_1 \to P)$,
3. a surjective submersion $\pi : P \to M$,

satisfying the following conditions:

1. the map $\pi : P \to M$ is $\mathcal{G}$-invariant,
2. the map $P \times_{\mathcal{G}_0,t} \mathcal{G}_1 \to P \times_{\pi,M,\pi} P$ given by $(p, \gamma) \mapsto (p, p \cdot \gamma)$ is a diffeomorphism.

Example (principal $\mathcal{G}$-bundle)

Let $\mathcal{G}$ be a Lie groupoid. Then, the structure map $t : \mathcal{G}_1 \to \mathcal{G}_0$ can be considered as a right principal $\mathcal{G}$-bundle.

“pseudo-functor” of principal $\mathcal{G}$-bundles

For a manifold M, let $B\mathcal{G}(M)$ denote the category, whose objects are principal $\mathcal{G}$-bundles over the manifold M, and whose morphisms are morphisms of principal $\mathcal{G}$-bundles over the manifold M. For similar reason as in the case of morphism of principal G-bundles over a manifold M, any morphism of principal $\mathcal{G}$-bundles over a manifold M is an isomorphism. Thus, the category $B\mathcal{G}(M)$ is a groupoid for each manifold M.

Consider the “psuedo-functor” $B\mathcal{G} : \text{Man} \to \text{Gpd}$, that assigns for each manifold M the category $B\mathcal{G}(M)$, and for each morphism $M \to M'$ the functor $B\mathcal{G}(M') \to B\mathcal{G}(M)$ given by pullback of principal bundles. It turns out that this psuedo-functor is “locally determined”.

Any psuedo-functor $\mathcal{D} : \text{Man} \to \text{Gpd}$ that is locally determined is called **a stack over (the site) Man**. We call $B\mathcal{G} : \text{Man} \to \text{Gpd}$ to be *the stack of principal $\mathcal{G}$-bundles*.

Notation and examples

Example (stack associated to a manifold)

Let M be a manifold and $(M \rightrightarrows M)$ be the associated Lie groupoid. This gives the stack $B(M \rightrightarrows M)$ over the site Man. By abuse of notation, we denote the stack $B(M \rightrightarrows M)$ as M or $\underline{M}$.

A stack $\mathcal{D} \to \text{Man}$ is said to be representable by a manifold if there exists a manifold M and an isomorphism of stacks $\mathcal{D} \cong B(M \rightrightarrows M) = M$.

Example (stack associated to an object)

Let $(\mathcal{C}, \mathcal{J})$ be a site. Further assume that $\mathcal{J}$ is subcanonical topology, that is, for each object U of $\mathcal{C}$, the functor $h_U : \mathcal{C}^{\text{op}} \to \text{Set}$ is a sheaf. Then, the functor $\mathcal{C}/U \to \mathcal{C}$ is a stack over the site $(\mathcal{C}, \mathcal{J})$. By abuse of notation, we denote the stack $\mathcal{C}/U \to \mathcal{C}$ by U or $\underline{U}$.

differentiable stacks

We are interested in stacks $\mathcal{D} : \text{Man} \to \text{Gpd}$ that are "representable by Lie groupoids"; in the sense that, there exists a Lie groupoid $\mathcal{G}$ such that $\mathcal{D} \cong B\mathcal{G}$. We call stacks $\text{Man} \to \text{Gpd}$ that are representable by Lie groupoids to be **differentiable stacks**.

Definition (atlas of a stack)

Let $\mathcal{D} \to \text{Man}$ be a stack. An *atlas for the stack* $\mathcal{D}$ is given by manifold X and a morphism of stacks $X \to \mathcal{D}$ such that,

- for each manifold Y and a morphism of stacks $Y \to \mathcal{D}$, the fiber product $X \times_{\mathcal{D}} Y$ is "represented by a manifold" and the projection $X \times_{\mathcal{D}} Y \to Y$ induces a surjective submersion at the level of manifolds.

Theorem

A stack $\mathcal{D} : Man \to Gpd$ is a differentiable stack if and only if there exists an atlas for $\mathcal{D} : Man \to Gpd$.

the morphism of stacks associated to morphism of Lie groupoids

Let $\mathcal{G} \to \mathcal{H}$ be a morphism of Lie groupoids. We want to associate a morphism of stacks $B\mathcal{G} \to B\mathcal{H}$.

Let M be a manifold and $P \to M$ be a principal $\mathcal{G}$-bundle over M; that is an object of $B\mathcal{G}(M)$. Consider the quotient space $(P \times_{\mathcal{G}_0} \mathcal{H}_1)/\mathcal{G}_1$ and the associated map $(P \times_{\mathcal{G}_0} \mathcal{H}_1)/\mathcal{G}_1 \to M$. The action of $\mathcal{H}$ on $\mathcal{H}_1$ induce an action of $\mathcal{H}$ on $(P \times_{\mathcal{G}_0} \mathcal{H}_1)/\mathcal{G}_1$ making $(P \times_{\mathcal{G}_0} \mathcal{H}_1)/\mathcal{G}_1 \to M$ into a principal $\mathcal{H}$-bundle. This gives the object level description of the functor $B\mathcal{G} \to B\mathcal{H}$, defined as $(P \to M) \mapsto ((P \times_{\mathcal{G}_0} \mathcal{H}_1)/\mathcal{G}_1 \to M)$.

We can define the functor $B\mathcal{G} \to B\mathcal{H}$ at the level of morphisms in a similar way. This is what we mean when we say morphism of stacks $B\mathcal{G} \to B\mathcal{H}$ associated to a morphism of Lie groupoids $\mathcal{G} \to \mathcal{H}$.

Question

Question

Does every morphism of stacks $B\mathcal{G} \to B\mathcal{H}$ come from a morphism of Lie groupoids?

No, for that, we need to allow a less-restricted version of a morphism of Lie groupoids. These are called bibundles (also called generalized morphisms of Lie groupoids).

As it turns out in the next couple of slides, any morphism of Lie groupoids gives a bibundle and any morphism of stacks “comes from” a bibundle.

a generalized morphism of Lie groupoids; $\mathcal{G} - \mathcal{H}$-bibundle

Definition

Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids. A $\mathcal{G} - \mathcal{H}$-*bibundle* consists of

1. a smooth manifold P,
2. an action of $\mathcal{G}$ on P given by $(a_{\mathcal{G}} : P \to \mathcal{G}_0, \mu : \mathcal{G}_1 \times_{s,\mathcal{G}_0} P \to P)$,
3. an action of $\mathcal{H}$ on P given by $(a_{\mathcal{H}} : P \to \mathcal{H}_0, \mu : P \times_{\mathcal{G}_0,t} \mathcal{H}_1 \to P)$

such that the following conditions are satisfied:

1. the action of $\mathcal{G}$ on P is compatible with the action of $\mathcal{H}$ on P,
2. the map $a_{\mathcal{H}} : P \to \mathcal{H}_0$ is a $\mathcal{G}$-equivariant map,
3. the map $a_{\mathcal{G}} : P \to \mathcal{G}_0$ is a principal $\mathcal{H}$-bundle.

Let $(F, f) : (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0) \to (\mathcal{H}_1 \rightrightarrows \mathcal{H}_0)$ be a morphism of Lie groupoids. The target map $t : \mathcal{H}_1 \to \mathcal{H}_0$ can be considered as a principal $\mathcal{H}$-bundle. The pullback of $t : \mathcal{H}_1 \to \mathcal{H}_0$ along the map $f : \mathcal{G}_0 \to \mathcal{H}_0$ gives the principal $\mathcal{H}$-bundle $\mathcal{G}_0 \times_{f,\mathcal{H}_0,t} \mathcal{H}_1 \to \mathcal{G}_0$.

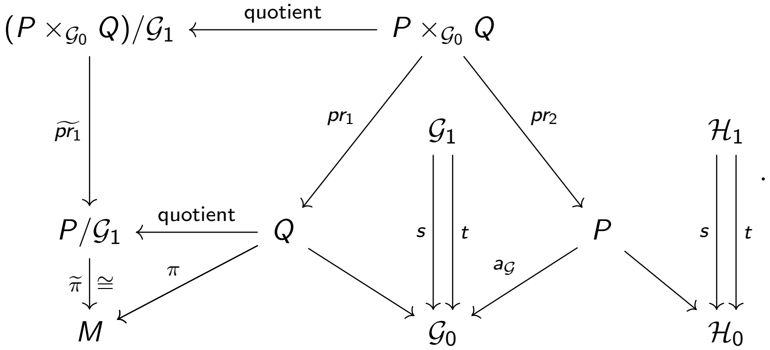
morphism of stacks associated to a bibundle

Let G, H be Lie groups and $\mathcal{G} \to \mathcal{H}$ be a morphism of Lie groups. Let $\pi : P \to M$ be a principal G-bundle. Then, the "associated fiber bundle construction" gives a principal H-bundle $(P \times H)/G \to M$. Thus, for each principal G-bundle over M, we get a principal H-bundle over M.

This construction can be imitated to assign a principal $\mathcal{H}$-bundle (over a manifold M) for each principal $\mathcal{G}$-bundle (over a manifold M) and a morphism of Lie groupoids $\mathcal{G} \to \mathcal{H}$. As mentioned before, every morphism of Lie groupoids $\mathcal{G} \to \mathcal{H}$ gives a $\mathcal{G} - \mathcal{H}$-bibundle. We do the above construction for a $\mathcal{G} - \mathcal{H}$-bibundle.

morphism of stacks associated to a bibundle

Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids. Let $Q : \mathcal{G} \to \mathcal{H}$ be a $\mathcal{G} - \mathcal{H}$-bibundle. Let M be a manifold and $\pi : P \to M$ be a principal $\mathcal{G}$-bundle over the manifold M. Then, $(P \times_{\mathcal{G}_0} Q)/\mathcal{G}_1 \to M$ can be considered as a principal $\mathcal{H}$-bundle. We see this as the following diagram,



Thus, we have a morphism of stacks $B\mathcal{G} \to B\mathcal{H}$.

Morita equivelent Lie groupoids

There are more than one way to declare when two Lie groupoids are Morita equivalent.

Definition ([1, Definition 3.40])

Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids. We say that $\mathcal{G}$ and $\mathcal{H}$ are *Morita equivalent* if there exists a $\mathcal{G} - \mathcal{H}$ bibundle $P : \mathcal{G} \to \mathcal{H}$ such that the action map $a_{\mathcal{G}} : P \to \mathcal{H}_0$ is a principal $\mathcal{G}$-bundle.

Remark

Two Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent if and only if the associated stacks $B\mathcal{G}$ and $B\mathcal{H}$ are isomorphic.

morphism of stacks associated to a Lie groupoid extension

Definition (Lie groupoid extension)

Let $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows M)$ and $\mathcal{H} = (\mathcal{H}_1 \rightrightarrows M)$ be Lie groupoids. A morphism of Lie groupoids $(F, 1_M) : (\mathcal{G}_1 \rightrightarrows M) \to (\mathcal{H}_1 \rightrightarrows M)$, where $F : \mathcal{G}_1 \to \mathcal{H}_1$ is a surjective submersion is called a *Lie groupoid extension*.

For every morphism of Lie groupoids $\mathcal{G} \to \mathcal{H}$, we can associate a $\mathcal{G} - \mathcal{H}$-bibundle. For each $\mathcal{G} - \mathcal{H}$- bibundle $P : \mathcal{G} \to \mathcal{H}$, we can associate a morphism of stacks $BP : \mathcal{G} \to B\mathcal{H}$. Combining these, for each morphism of Lie groupoids $\mathcal{G} \to \mathcal{H}$, we can associate a morphism of stacks $B\mathcal{G} \to B\mathcal{H}$. In particular, for a Lie groupoid extension $(F, 1_M) : \mathcal{G} \to \mathcal{H}$, we have the associated morphism of stacks $F : B\mathcal{G} \to B\mathcal{H}$.

The special nature of morphism of Lie groupoids $(F, 1_M) : \mathcal{G} \to \mathcal{H}$ reflect in the morphism of stacks $F : B\mathcal{G} \to B\mathcal{H}$. For example, the morphism $F : B\mathcal{G} \to B\mathcal{H}$ is an epimorphism of stacks (which we will see soon).

Definition (epimorphism of stacks)

Let $F : \mathcal{D} \to \mathcal{C}$ be a morphism of stacks. We say that F is an epimorphism if for each object U of Man and a morphism of stacks $q : U \to \mathcal{C}$, there exists a surjective submersion $f : V \to U$ and a morphism of stacks $p : V \to \mathcal{D}$ with a 2-commutative diagram,

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & U \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ \mathcal{D} & \xrightarrow{\quad F \quad} & \mathcal{C} \end{array} .$$

Equivalently (using 2-Yoneda Lemma), a morphism $F : \mathcal{D} \to \mathcal{C}$ is an *epimorphism of stacks*, if, for each manifold U and an object b of $\mathcal{C}(U)$, there exists a surjective submersion $f : V \to U$ and an object a of $\mathcal{D}(V)$ such that $F(a) \cong f^*(b)$, where f^* is the morphism $\mathcal{C}(U) \to \mathcal{C}(V)$.

representable surjective submersion

For our purposes, on the morphism of stacks we will come across later, we would like to impose a stronger condition than that of an epimorphism of stacks.

Definition (representable surjective submersion)

Let $F : \mathcal{D} \to \mathcal{C}$ be a morphism of stacks. We say that F is a *represetable surjective submersion* if for each object U of Man and a morphism of stacks $q : U \to \mathcal{C}$, the 2-fiber product $\mathcal{D} \times_{\mathcal{C}} U$ is "reprersentable by a manifold" and the projection map $\mathcal{D} \times_{\mathcal{C}} U \to U$ is a surjective submersion at the level of manifolds.

As it is clear from the definitions, any representable surjective submersion is an epimorhism of stacks.

gerbe over a stack as a morphism of stacks

Definition (a gerbe over a stack)

Let $\mathcal{D}$ and $\mathcal{C}$ be differentiable stacks. A morphism of stacks $\mathcal{D} \to \mathcal{C}$ is said to be *a gerbe over the stack* $\mathcal{C}$ if the morphism $F : \mathcal{D} \to \mathcal{C}$ and the diagonal morphism $\Delta_F : \mathcal{D} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{D}$ are epimorphism of stacks.

We have already mentioned what it means for a morphism to be an epimorphism of stacks. As any surjective submersion $f : M \to N$ is a map of local sections[1], equivalent condition of epimorphism can be rephrased as follows:

A morphism of stcks $F : \mathcal{D} \to \mathcal{C}$ is an epimorphism of stacks if for each manifold U and an object b of $\mathcal{C}(U)$, there exists an open cover $\{U_\alpha\}$ of U and objects $a_\alpha \in \mathcal{D}(U_\alpha)$ such that $F(a_\alpha) \cong b|_{U_\alpha}$ in $\mathcal{C}(U_\alpha)$ for each α.

[1] there exists an open cover $\{U_\alpha\}$ of N and sections $s_\alpha : U_\alpha \to M$ of $f : M \to N$

$F : B\mathcal{G} \to B\mathcal{H}$ maps the trivial principal bundle on $\mathcal{G}$ to the trivial principal bundle on $\mathcal{H}$

Lemma ([4, Lemma 5.5])

Let $F : B\mathcal{G} \to B\mathcal{H}$ be the morphism of stacks associated to a Lie groupoid extension $F : (\mathcal{G}_1 \rightrightarrows M) \to (\mathcal{H}_1 \rightrightarrows M)$. Then, F maps the principal $\mathcal{G}$-bundle $t : \mathcal{G}_1 \to M$ to the principal $\mathcal{H}$-bundle $t : \mathcal{H}_1 \to M$. Furtehr, F maps the pullback of $t : \mathcal{G}_1 \to M$ along a map $f : U \to M$ to the pullback of $t : \mathcal{H}_1 \to M$ along the map $f : U \to M$.

$F : B\mathcal{G} \to B\mathcal{H}$ is an epimorphism

Let U be an object of Man and $q : U \to B\mathcal{H}$ be a morphism of stacks. By 2-Yoneda lemma, this assigns a principal $\mathcal{H}$-bundle over the manifold U, say $\pi : Q \to U$.

By "local triviality" of principal bundle, there exists an open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of U and smooth maps $U_\alpha \to M$ such that, the principal bundle $\pi^{-1}(U_\alpha) \to U_\alpha$ is the pullback of $t : \mathcal{H}_1 \to M$ along the smooth map $\sigma_\alpha : U_\alpha \to M$ for each $\alpha \in \Lambda$.

Consider the pullback of $t : \mathcal{G}_1 \to M$ along the smooth map $\sigma_\alpha : U_\alpha \to M$. This gives a principal $\mathcal{G}$-bundle over U_α, that is a morphism of stacks $U_\alpha \to B\mathcal{G}$. By above lemma, F takes the pullback of $t : \mathcal{G}_1 \to M$ along $\sigma_\alpha : U_\alpha \to M$ to the pullback of $t : \mathcal{H}_1 \to M$ along $\sigma_\alpha : U_\alpha \to M$. This gives the desired 2-commutative diagram. Thus, $F : B\mathcal{G} \to B\mathcal{H}$ is an epimorphim of stacks.

an alternate description of the assignment $\mathcal{G} \to B\mathcal{G}$

Given a Lie groupoid $\mathcal{G}$, consider the pseudo-functor $h_{\mathcal{G}} : \text{Man}^{op} \to \text{Gpd}$ defined as $M \mapsto \text{Mor}_{\text{LieGpd}}((M \rightrightarrows M), \mathcal{G})$. This pseudo-functor $h_{\mathcal{G}}$ is not in general "locally determined"; that is not a stack in general. It turns out that the stackification of the pseudo-functor $h_{\mathcal{G}} : \text{Man}^{op} \to \text{Gpd}$ is the pseudo-functor $B\mathcal{G} : \text{Man}^{op} \to \text{Gpd}$; the stack of principal $\mathcal{G}$-bundles mentioned above ([5] Corollary I.2.2).

As $(F, 1_M) : (\mathcal{G}_1 \rightrightarrows M) \to (\mathcal{H}_1 \rightrightarrows M)$ is a Lie groupoid extension, the fibered product $\mathcal{G} \times_{\mathcal{H}} \mathcal{G}$ is a Lie groupoid. As Yoneda embedding and stackification preserves fibered product, we have

$$B(\mathcal{G} \times_{\mathcal{H}} \mathcal{G}) \cong B\mathcal{G} \times_{B\mathcal{H}} B\mathcal{G}.$$

some results

Lemma ([4, Lemma 5.8.])

Let $\mathcal{G} \to \mathcal{H}$ be a Lie groupoid extension. Then, the fibered product $\mathcal{G} \times_{\mathcal{H}} \mathcal{G}$ is a transitive Lie groupoid.

Lemma ([4, Lemma 5.9])

Any transitive Lie groupoid $\mathcal{G}$ is Morita equivalent to the isotropy group $\mathcal{G}_x$ for any $x \in \mathcal{G}_0$, that is, the Lie groupoid $(\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ is Morita equivalent to the Lie groupoid $(\mathcal{G}_x \rightrightarrows *)$.

$\Delta_F : B\mathcal{G} \to B\mathcal{G} \times_{B\mathcal{H}} B\mathcal{G}$ is an epimorphism

Combining above lemmas, we see that $\mathcal{G} \times_{\mathcal{H}} \mathcal{G}$ is Morita equivalent to a Lie groupoid of the form $(K \rightrightarrows *)$. Thus, the stacks $B(\mathcal{G} \times_{\mathcal{H}} \mathcal{G})$ and $B(K \rightrightarrows *)$ are isomorphic. So, the diagonal morphism $\Delta_F \colon B\mathcal{G} \to B\mathcal{G} \times_{B\mathcal{H}} B\mathcal{G}$ is isomorphic to the map $B\mathcal{G} \to B(K \rightrightarrows *)$.

Using a similar argument as before, we see that for any morphism of Lie groupoids $(\mathcal{G}_1 \rightrightarrows M) \to (K \rightrightarrows *)$, the corresponding morphism of stacks $B\mathcal{G} \to B\mathcal{K}$ is an epimorphism of stacks. Thus, $\Delta_F \colon B\mathcal{G} \to B\mathcal{G} \times_{B\mathcal{H}} B\mathcal{G}$ is an epimorphism of stacks.

Proposition

Given a Lie groupoid extension $\phi \colon (\mathcal{G}_1 \rightrightarrows M) \to (\mathcal{H}_1 \rightrightarrows M)$ the diagonal morphism of stacks $\Delta_F \colon B\mathcal{G} \to B\mathcal{G} \times_{B\mathcal{H}} B\mathcal{G}$ is an epimorphism of stacks.

First main theorem

Theorem

1. *Given a Lie groupoid extension* $\phi\colon (\mathcal{G}_1 \rightrightarrows M) \to (\mathcal{H}_1 \rightrightarrows M)$, *the corresponding morphism of stacks* $F\colon B\mathcal{G} \to B\mathcal{H}$ *is a gerbe over the stack* $B\mathcal{H}$.
2. *Let* $\phi\colon \mathcal{G}_1 \to \mathcal{H}_1 \rightrightarrows M$ *and* $\phi''\colon \mathcal{G}''_1 \to \mathcal{H}''_1 \rightrightarrows M''$ *be Morita equivalent Lie groupoid extensions. Let* $\Phi\colon B\mathcal{G} \to B\mathcal{H}$ *and* $\Phi''\colon B\mathcal{G}'' \to B\mathcal{H}''$ *be the respective morphism of stacks corresponding to* ϕ *and* ϕ''. *Then,* Φ *and* Φ'' *are isomorphic. in the sense that, the following diagram is 2-commutative,*

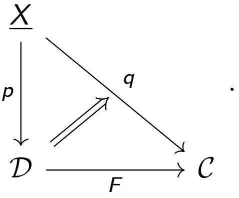
$$\begin{array}{ccc} B\mathcal{G}'' & \xrightarrow{\Phi''} & B\mathcal{H}'' \\ \cong\downarrow & \nearrow & \downarrow\cong \\ B\mathcal{G} & \xrightarrow{\Phi} & B\mathcal{H}. \end{array}$$

existence of a special kind of atlas for $\mathcal{C}$

Let $F : \mathcal{D} \to \mathcal{C}$ be a gerbe over the differentiable stack $\mathcal{C}$. As F is an epimorphism, for every manifold Y and a morphism of stacks $q : Y \to \mathcal{C}$, there exists a surjective submersion $g : X \to Y$ and a morphism of stacks $p : X \to \mathcal{D}$ with a 2-commutative diagram given by $F \circ p \Rightarrow q \circ g$.

Choose $q : V \to \mathcal{C}$ to be an atlas for $\mathcal{C}$. As $g : U \to V$ is a surjective submersion, the composition $q \circ g : U \to \mathcal{C}$ is an atlas for $\mathcal{C}$.

Thus, for $F : \mathcal{D} \to \mathcal{C}$, there exists an atlas $q : X \to \mathcal{C}$ for $\mathcal{C}$ and a morphism of stacks $p : U \to \mathcal{D}$ as in the following diagram:



the morphism of stacks $p : X \to \mathcal{D}$ is an epimorphism

Let M be a manifold and $r\colon \underline{M} \to \mathcal{D}$ be a morphism of stacks.

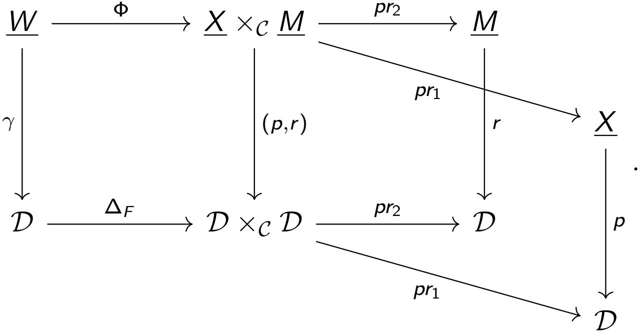
As $F \circ p = q : \underline{X} \to \mathcal{C}$ is an atlas for $\mathcal{C}$, for the morphism $F \circ r : \underline{M} \to \mathcal{C}$, the fibered product $\underline{X} \times_{\mathcal{C}} \underline{M}$ is representable by a manifold.

Since the diagonal morphism $\Delta_F\colon \mathcal{D} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{D}$ is an epimorphism of stacks, for the morphism of stacks $(p, r)\colon \underline{X} \times_{\mathcal{C}} \underline{M} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{D}$, there exists a manifold W and a surjective submersion $\Phi\colon \underline{W} \to \underline{X} \times_{\mathcal{C}} \underline{M}$ producing the following 2-commutative diagram,

$$\begin{array}{ccc} \underline{W} & \xrightarrow{\quad \Phi \quad} & \underline{X} \times_{\mathcal{C}} \underline{M} \\ {\scriptstyle \gamma}\big\downarrow & \nearrow\!\!\!\!\nearrow & \big\downarrow{\scriptstyle (p,r)} \\ \mathcal{D} & \xrightarrow{\quad \Delta_F \quad} & \mathcal{D} \times_{\mathcal{C}} \mathcal{D} \end{array} .$$

the morphism of stacks $p : X \to \mathcal{D}$ is an epimorphism

Extending the above Diagram along the first projections and second projections obtain the following 2-commutative diagram,



the morphism of stacks $p : X \to \mathcal{D}$ is an epimorphism

The composition $pr_2 \circ \Phi \colon \underline{W} \to \underline{M}$ is a surjective submersion. This gives the following diagram of morphism of stacks,

$$
\begin{array}{ccc}
\underline{W} & \xrightarrow{pr_2 \circ \Phi} & \underline{M} \\
{\scriptstyle pr_1 \circ \Phi}\big\downarrow & & \big\downarrow{\scriptstyle r} \\
\underline{X} & \xrightarrow{p} & \mathcal{D}
\end{array} .
$$

Thus, the morphism of stacks $p \colon \underline{X} \to \mathcal{D}$ associated to the atls $q : X \to \mathcal{C}$ in previous diagram is an epimorphism of stacks.

the morphism $p\colon \underline{X} \to \mathcal{D}$ is an atlas for $\mathcal{D}$

We prove that, (under certain mild assumption on the diagonal morphism $\Delta_F : \mathcal{D} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{D}$) the morphism $p : \underline{X} \to \mathcal{D}$ is an atlas for $\mathcal{D}$. We borrow a result from the paper [3].

Lemma ([3, Proposition 2.6])

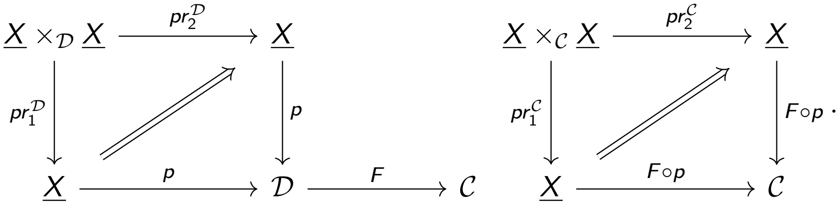
A morphism of stacks $r\colon \underline{X} \to \mathcal{D}$ is an atlas for $\mathcal{D}$ if

1. *the morphism $r\colon \underline{X} \to \mathcal{D}$ is an epimorphism of stacks,*
2. *the fibered product $\underline{X} \times_{\mathcal{D}} \underline{X}$ is representable by a manifold and that the projection maps $pr_1\colon X \times_{\mathcal{D}} X \to X$ and $pr_2\colon X \times_{\mathcal{D}} X \to X$ are submersions.*

We have already proved that $p : \underline{X} \to \mathcal{D}$ is an epimorphism of stacks. It only remains to prove $p : \underline{X} \to \mathcal{D}$ satisfies the second condition. To prove this, we assume that, the diagonal morphism $\mathcal{D} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{D}$ is a representable surjective submersion. This assures that $p : \underline{X} \to \mathcal{D}$ satisfies the condition 2 and we conlcude that $p : \underline{X} \to \mathcal{D}$ is an atlas for $\mathcal{D}$.

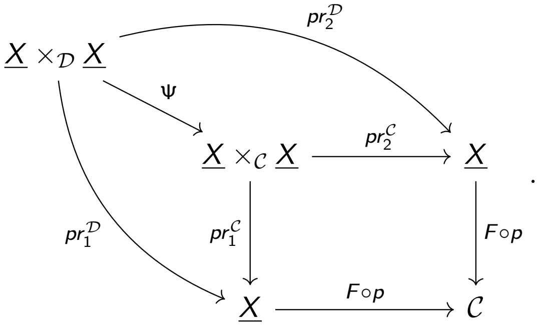
the morphism $p\colon \underline{X} \to \mathcal{D}$ is an atlas for $\mathcal{D}$

For $p\colon \underline{X} \to \mathcal{D}$ and for $F \circ p\colon \underline{X} \to \mathcal{C}$, we have following pull back diagrams,



the morphism $p\colon \underline{X} \to \mathcal{D}$ is an atlas for $\mathcal{D}$

By uniqueness of pullback, there exists a unique morphism of stacks $\Psi\colon \underline{X} \times_{\mathcal{D}} \underline{X} \to \underline{X} \times_{\mathcal{C}} \underline{X}$ with following 2-commutative diagram,



$p\colon \underline{X} \to \mathcal{D}$ is an atlas for $\mathcal{D}$

Since the diagonal morphism $\Delta_F\colon \mathcal{D} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{D}$ is a representable surjective submersion, the morphism of stacks $(p,p)\colon \underline{X} \times_{\mathcal{C}} \underline{X} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{D}$ produces the following 2-fiber product diagram,

$$\begin{array}{ccc}
\mathcal{D} \times_{\mathcal{D}\times_{\mathcal{C}}\mathcal{D}} (\underline{X} \times_{\mathcal{C}} \underline{X}) & \xrightarrow{\;pr_2\;} & \underline{X} \times_{\mathcal{C}} \underline{X} \\
{\scriptstyle pr_1}\big\downarrow & \nearrow\!\!\!\nearrow & \big\downarrow{\scriptstyle (p,p)} \\
\mathcal{D} & \xrightarrow{\;\Delta_F\;} & \mathcal{D} \times_{\mathcal{C}} \mathcal{D}
\end{array} .$$

The isomorphism $\mathcal{D} \times_{\mathcal{D}\times_{\mathcal{C}}\mathcal{D}} (\underline{X} \times_{\mathcal{C}} \underline{X}) \cong \underline{X} \times_{\mathcal{D}} \underline{X}$ ([7, Corollary 69]) lets us identify the morphisms of stacks $pr_2\colon \mathcal{D} \times_{\mathcal{D}\times_{\mathcal{C}}\mathcal{D}} (\underline{X} \times_{\mathcal{C}} \underline{X}) \to \underline{X} \times_{\mathcal{C}} \underline{X}$ and $\Psi\colon \underline{X} \times_{\mathcal{D}} \underline{X} \to \underline{X} \times_{\mathcal{C}} \underline{X}$.

$p\colon \underline{X} \to \mathcal{D}$ is an atlas for $\mathcal{D}$

Thus, we have the following 2-fiber product diagram,

$$\begin{array}{ccc}
\underline{X} \times_{\mathcal{D}} \underline{X} & \xrightarrow{\Psi} & \underline{X} \times_{\mathcal{C}} \underline{X} \\
\downarrow & \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle (p,p)} \\
\mathcal{D} & \xrightarrow{\Delta_F} & \mathcal{D} \times_{\mathcal{C}} \mathcal{D}
\end{array} .$$

As the diagonal morphism $\Delta_F\colon \mathcal{D} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{D}$ is a representable surjective submersion, the 2-fiber product $\underline{X} \times_{\mathcal{D}} \underline{X}$ is representable by a manifold and $\Psi\colon \underline{X} \times_{\mathcal{D}} \underline{X} \to \underline{X} \times_{\mathcal{C}} \underline{X}$ is a surjective submersion. As compositions of surjective submersions, $pr_1^{\mathcal{D}} = pr_1^{\mathcal{C}} \circ \Psi$ and $pr_2^{\mathcal{D}} = pr_2^{\mathcal{C}} \circ \Psi$ are surjective submersions as well. Thus, $p\colon \underline{X} \to \mathcal{D}$ is an atlas for $\mathcal{D}$.

some results

Lemma ([4, Lemma 4.8.])

Let $F\colon \mathcal{D} \to \mathcal{C}$ be a gerbe over a stack. Assume that the diagonal morphism $\Delta_F\colon \mathcal{D} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{D}$ is a representable surjective submersion. Then the morphism of stacks $F\colon \mathcal{D} \to \mathcal{C}$ gives a Lie groupoid extension $\mathcal{G}_p = (X \times_{\mathcal{D}} X \rightrightarrows X) \to \mathcal{G}_q = (X \times_{\mathcal{C}} X \rightrightarrows X)$, where $p\colon \underline{X} \to \mathcal{D}$ and $q\colon \underline{X} \to \mathcal{C}$ are atlases as mentioned above.

Lemma ([4, Lemma 4.9.])

Let $F\colon \mathcal{D} \to \mathcal{C}$ be a gerbe over a stack. Assume that the diagonal morphism $\Delta_F\colon \mathcal{D} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{D}$ is a representable surjective submersion. Then, up to a Morita equivalence, the Lie groupoid extension in Labove lemma does not depend on the choice of $q\colon \underline{X} \to \mathcal{C}$.

Second main theorem

Theorem ([4, Theorem 4.10.])

*Let $F\colon \mathcal{D} \to \mathcal{C}$ be a gerbe over a stack. Assume that the diagonal morphism $\Delta_F\colon \mathcal{D} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{D}$ is a representable surjective submersion. Then there exists a Lie groupoid extension $\phi\colon \mathcal{G} \to \mathcal{H}$ such that the corresponding morphism of stacks $\Phi\colon B\mathcal{G} \to B\mathcal{H}$ (Lemma **??**) satisfies the following 2-commutative diagram,*

$$\begin{array}{ccc} B\mathcal{G} & \xrightarrow{\Phi} & B\mathcal{H} \\ {\scriptstyle\cong}\downarrow & \nearrow\!\!\!\!\nearrow & \downarrow{\scriptstyle\cong} \\ \mathcal{D} & \xrightarrow{F} & \mathcal{C} \end{array} .$$

Main remark I

Remark ([4, Remark 4.11.])

Observe that we have not made full use of the condition Δ_F being a surjective submersion. We have only used the following. The morphism of stacks $p\colon \underline{X} \to \mathcal{D}$ is such that, $\underline{X} \times_{\mathcal{D}} \underline{X}$ is representable by a manifold and the morphism of stacks $\Psi\colon \underline{X} \times_{\mathcal{D}} \underline{X} \to \underline{X} \times_{\mathcal{C}} \underline{X}$ is a surjective submersion at the level of manifolds.

Main remarks II

Remark ([4, Remark 5.12.])

Let $\mathcal{D} \to \mathcal{C}$ be a gerbe over a stack. Assume further that $\mathcal{D} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{D}$ is a representable surjective submersion. In particular, this means there exists atlases $\underline{X} \to \mathcal{C}$ and $\underline{X} \to \mathcal{D}$ respectively for the stacks $\mathcal{C}$ and $\mathcal{D}$ such that the smooth map $X \times_{\mathcal{D}} X \to X \times_{\mathcal{C}} X$ is a surjective submersion. Now we make the following observation:
Let $(\Phi, 1_M) \colon (\mathcal{G}_1 \rightrightarrows M) \to (\mathcal{H}_1 \rightrightarrows M)$ be a Lie groupoid extension and $B\mathcal{G} \to B\mathcal{H}$ be the associated morphism of stacks. Then

1. *the morphism of stacks $B\mathcal{G} \to B\mathcal{H}$ is a gerbe over the stack $B\mathcal{H}$.*
2. *there exists atlases $\underline{M} \to B\mathcal{H}$ and $\underline{M} \to B\mathcal{G}$ satisfying $\underline{M} \times_{B\mathcal{G}} \underline{M} = \mathcal{G}_1$ and $\underline{M} \times_{B\mathcal{H}} \underline{M} = \mathcal{H}_1$. Moreover, the smooth map $\Phi \colon \mathcal{G}_1 \to \mathcal{H}_1$ associated to the morphism of stacks $\underline{M} \times_{B\mathcal{G}} \underline{M} \to \underline{M} \times_{B\mathcal{H}} \underline{M}$ is a surjective submersion.*

concluding statement

One can now use this correspondece to understand the notion of connections and other structrues on gerbes over stacks using the set up of [3] and [2], defined as morphisms of stacks and morphisms of Lie groupoids respectively.

Thanks for your attention.

References

- Eugene Lerman, *Orbifolds as stacks?* Enseign. Math. (2), 56(3-4), (2010), Pages 315 – 363.
- Camille Laurent-Gengoux, Mathieu Stiénon, Ping Xu, *Non-abelian differentiable gerbes*, Advances in Mathematics, Volume 220, Issue 5, 2009, Pages 1357 – 1427.
- Kai Behrend, Ping Xu, *Differentiable stacks and gerbes*. J. Symplectic Geom. 9 (2011), no. 3, Page 285 – 341.
- Saikat Chatterjee, Praphulla Koushik, *On two notions of a gerbe over a stack*, Bulletin des Sciences Mathématiques, Volume 163, 2020.
- David Carchedi, Categorical Properties of Topological and Differentiable Stacks.
- Kirill C. H. Mackenzie, General theory of Lie groupoids and Lie algebroids, Cambridge University Press 2005.
- David Metzler, Topological and Smooth Stacks.