Stacky Lie algebroids

Miquel Cueca Ten
(joint with Daniel Álvarez)

Georg-August-Universität
Göttingen, Germany

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Motivation

$\mathbb{LG} = \{\text{Lie groupoids category}\}, \ \mathbb{LA} = \{\text{Lie algebroids category}\}$
and we have the Lie functor between them

$$\textbf{Lie} : \mathbb{LG} \to \mathbb{LA},$$

but **Lie** is not essentially surjective Molino, Weinstein...
(Explicit obstructions computed by Crainic-Fernandes).

What is the integration of a general Lie algebroid?

Answer Tseng-Zhu: an étale stacky Lie groupoid.

What is the infinitesimal counterpart of a stacky Lie groupoid?

A stacky Lie algebroid. UFO!!
Moreover we want the following diagram to commute

$$\begin{array}{ccc} \mathbb{LG} & \xrightarrow{\textbf{Lie}} & \mathbb{LA} \\ \cap & & \cap \\ \mathbb{SLG} & \xrightarrow{\textbf{SLie}} & \mathbb{SLA} \end{array}$$

Structure of the talk

Goal

Introduce and study **stacky Lie algebroids**

(A) Two related problems:
  1. Stacky Lie algebroids.
  2. Lie algebroids over a differentiable stack.

(B) Defining stacky Lie algebroids.

(C) Main properties.

(D) Work in progress:
  1. **SLie** functor.
  2. Integration of Courant algebroids.

Part A

Two related problems:

Stacky Lie algebroids and Lie algebroids over a differentiable stack.

Differentiable stacks

Grothendieck, Behrend-Xu

Idea: Differentiable stacks are good models for singular spaces.

$\mathbb{M}\mathrm{an} = \{\text{Smooth manifolds category}\}$ with the Grothendieck topology given by open covers.

Differentiable stack: A category $\mathfrak{X}$ with a functor $F_{\mathfrak{X}} : \mathfrak{X} \to \mathbb{M}\mathrm{an}$ satisfying:

1. Pullbacks exists and are "universal".
2. Gluing properties (sheaves like conditions).
3. It has a presentation $p : X \to \mathfrak{X}$.

Differentiable stacks form a 2-category:

- Objects: Differentiable stacks $\mathfrak{X}, \mathfrak{J}$...
- 1-morphisms: Functors $F : \mathfrak{X} \to \mathfrak{J}$ s.t. $F_{\mathfrak{J}} F = F_{\mathfrak{X}}$.
- 2-morphisms: Natural trans $\eta : F \Rightarrow F'$ s.t. $F_{\mathfrak{J}}\eta(x) = \mathrm{id}_{F_{\mathfrak{X}}(x)}$.

The dictionary

Given $G \rightrightarrows M$ a Lie groupoid define the category $[M/G]$:

- Objects: Principal right G-bundles.
- Morphism: morphisms of principal G-bundles.

Theorem Behrend-Xu:

1. For any lie groupoid $G \rightrightarrows M$, $[M/G]$ is a differentiable stack.
2. Given a differentiable stack with a presentation $p : X \to \mathfrak{X}$, $X \times_{\mathfrak{X}} X \rightrightarrows X$ is a Lie groupoid and $\mathfrak{X} \cong [X/X \times_{\mathfrak{X}} X]$.
3. $G \rightrightarrows M$ and $H \rightrightarrows N$ Morita equivalent iff $[M/G] \cong [N/H]$.

Remark:The dictionary can be extended to 1- and 2-morphisms.

Conclusion

Differentiable stacks can be thought as the Morita class of a Lie groupoid.

Stacky Lie groupoids

Tseng-Zhu

A **Stacky Lie groupoid** $\mathfrak{G} \rightrightarrows M$ consist of a differentiable stack $\mathfrak{G}$ and a manifold M together with

- 1-morphisms: $s, t, m, 1, i$.
- 2-isomorphisms: $\alpha : m(\mathrm{id} \times m) \Rightarrow m(m \times \mathrm{id})$,
  $u_l : m\langle 1t, \mathrm{id}\rangle \Rightarrow \mathrm{id}$, $u_r : m\langle \mathrm{id}, 1s\rangle \Rightarrow \mathrm{id}$,
  $\iota_l : m\langle i, \mathrm{id}\rangle \Rightarrow 1s$, $\iota_r : m\langle \mathrm{id}, i\rangle \Rightarrow 1t$.

satisfying

1. $s1 = \mathrm{id}_M$, $t1 = id_M$, $si = t$, $ti = s$, $sm = sn_s$, $tm = tn_t$.
2. Higher coherence conditions for the 2-isomorphism.

Why? Integrate Lie algebroids Tseng-Zhu, symmetries of stacks Blohmann, actions on differentiable stacks Bursztyn-Noseda-Zhu, hamiltonian actions on symplectic stacks Hoffman-Sjamaar-Zhu...

Examples

- Any Lie groupoid $G \rightrightarrows M$.
- If $H \rightrightarrows G$ is a strict 2-groups then $[G/H] \rightrightarrows *$ is a stacky Lie group. More generally, given

$$\begin{array}{ccc} D & \rightrightarrows & K \\ \downarrow\downarrow & & \downarrow\downarrow \\ M & \rightrightarrows & M \end{array}$$

  a double Lie groupoid covering the identity groupoid then $[K/D] \rightrightarrows M$ is a stacky Lie groupoid.
- Tseng-Zhu: For any Lie algebroid $A \to M$ the Weinstein groupoid $\mathcal{W}(A) \rightrightarrows M$ is a stacky Lie groupoid.

Guiding Theorem (Zhu)

There is a 1-1 correspondence between stacky Lie groupoids and Lie 2-groupoids modulo 1-Morita equivalence.

Problem 1: Stacky Lie algebroids

1. **Strategy 1** : Define vector bundle $\mathfrak{E} \to M$ and a Lie algebroid structure there. Waldron, still work in progress!!

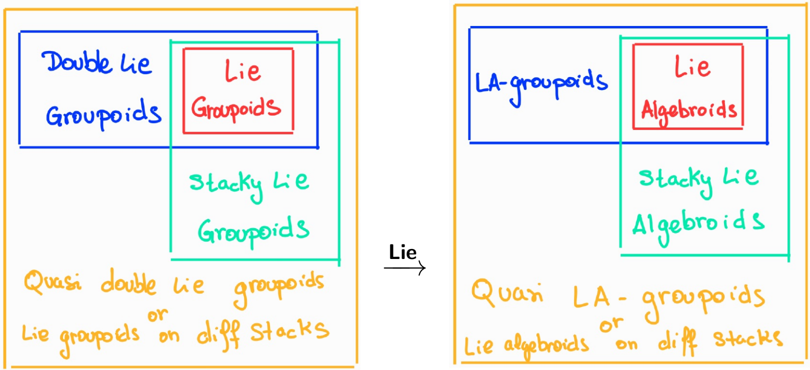
2. **Strategy 2**: Using the dictionary re-interpret stacky Lie groupoids as a double structure and differenciate.

$$\begin{array}{ccccc} G & \rightrightarrows & X & \to & \mathfrak{G} \\ ? & & ? & & \downarrow\downarrow \\ M & \rightrightarrows & M & \to & M \end{array}$$

Then a stacky Lie algebroid will be the "Morita" class of one of such structures.

Problem 2: Lie algebroids over a differentiable stack

Including double groupoids and applying the Lie functor



$$\begin{array}{ccccc} D & \rightrightarrows & K & \longrightarrow & [K/D] \\ \Downarrow & & \Downarrow & & \Downarrow \\ G & \rightrightarrows & M & \longrightarrow & [M/G] \end{array} \qquad \begin{array}{ccccc} A_D & \rightrightarrows & A_K & \longrightarrow & [A_K/A_D] \\ \Downarrow & & \Downarrow & & \Downarrow \\ G & \rightrightarrows & M & \longrightarrow & [M/G] \end{array}$$

If $G = M \rightsquigarrow$ "Stacky side", strict arrows $\rightsquigarrow$ "Double side".

Part B

Defining stacky Lie algebroids

via quasi LA-groupoids

Semistric Lie 2-algebras

Baez-Crans

If $G = M = \{*\}$ qLA-groupoids already appear in the literature.

A **semistrict Lie 2-algebra** $L \equiv L_1 \rightrightarrows L_0$ is a 2 vector space endowed with

1. A skew-symmetric bilinear functor $[\cdot,\cdot] : L \times L \to L$
2. An antisymmetric trilinear natural isomorphism $\alpha_{x,y,z} : [[x,y],z] \Rightarrow [x,[y,z]] + [[x,z],y]$.

satisfying

$$\alpha_{[w,x],y,z}\Big([\alpha_{w,x,z}, y] + 1\Big)\Big(\alpha_{w,[x,z],y} + \alpha_{[w,z],x,y} + \alpha_{w,x,[y,z]}\Big) = [\alpha_{w,x,y}, z]\,\Big(\alpha_{[w,y],x,z} + \alpha_{w,[x,y],z}\Big)\Big([\alpha_{w,y,z}, x] + 1\Big)\Big([w, \alpha_{x,y,z}] + 1\Big)$$

Why it works?

Theorem Baez-Crans: There is a 1-1 correspondence between semistrict Lie 2-algebras and 2-term L_∞-algebras.

Graded manifolds perspective

Replacing a 2 vector space by 2 vector bundle $\rightsquigarrow$ VB-groupoid.

Problem: for Lie algebroids, $[\cdot,\cdot]$ is not a functor!!

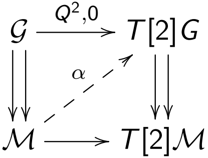
Theorem Vaintrob: There is a 1-1 correspondence between Lie algebroids and degree 1 Q-manifolds.

Theorem Mehta: There is a 1-1 correspondece between VB-groupoids and degree 1 Lie groupoids. Moreover, LA-groupoids are in correspondence with degree 1 Q-groupoids.

Definition Mehta A degree n Q-groupoid $(\mathcal{G} \rightrightarrows \mathcal{M}, Q)$ is a Lie groupoid object between degree n manifolds endowed with $Q \in \mathfrak{X}^1(\mathcal{G})$ such that

$$Q^2 = 0 \quad \textit{and} \quad Q : \mathcal{G} \to T[1]\mathcal{G} \text{ groupoid morphism.}$$

Quasi Q-groupoids



Definition 1

A degree n quasi Q-groupoid $(\mathcal{G} \rightrightarrows \mathcal{M}, Q, \alpha)$ is a Lie groupoid object in the category of degree n manifolds, $Q \in \mathfrak{X}^1(\mathcal{G})$ and $\alpha \in \Gamma_2 \mathcal{A}_{\mathcal{G}}$ such that $Q : \mathcal{G} \to T[1]\mathcal{G}$ groupoid morphism and

$$Q^2 = \alpha^l - \alpha^r \quad [Q, \alpha^l] = 0.$$

$(F, R) : (\mathcal{G} \rightrightarrows \mathcal{M}, Q, \alpha) \to (\mathcal{G}' \rightrightarrows \mathcal{M}', Q', \alpha')$ is a morphism if

1. $F : \mathcal{G} \to \mathcal{G}'$ groupoid morphism.
2. $R : T[1]F \circ Q \Rightarrow Q' \circ F$ natural transformation such that

$$T[1]F \circ \widehat{\alpha} = (\widehat{\alpha}' \circ F)R(R \circ Q)$$

Classical definition

Given

$$\begin{array}{ccc} H & \rightrightarrows & E \\ \downarrow & & \downarrow \\ G & \rightrightarrows & M \end{array}$$

a VB-groupoid then $H[1] \rightrightarrows E[1]$ defines a degree 1 groupoid.

Definition 2

A quasi LA-groupoid structure on $(H \rightrightarrows E; G \rightrightarrows M)$ is a quasi Q-groupoid structure on $H[1] \rightrightarrows E[1]$.

That means that $H \to G$ carries an anchor and a bracket but also

- A new 3-bracket that controls Jacobi.
- A new 2-anchor that controls the anchor being bracket preserving.

many equations!!

Particular case

If the base groupoid is $M \rightrightarrows M$ (this is the case that we need) we have:

Proposition 1

A qLA-groupoid on $(H \rightrightarrows E; M \rightrightarrows M)$ is equivalent to an almost Lie algebroid $(H \to M, [\cdot,\cdot], a)$ and $\phi : \wedge^3 E \to C$ satisfying:

1. $graph(m_H) \to graph(m_M)$ is an almost Lie subalgebroid of $H \times H \times H \to M \times M \times M$.
2. $Jac_{[\cdot,\cdot]}(h_0, h_1, h_2) = (\phi^l - \phi^r)(h_0, h_1, h_2)$.
3. $a(\phi^l(h_0, h_1, h_2)) = 0$.
4. $[\phi^l(h_0,h_1,h_2),h_3]+[\phi^l(h_2,h_3,h_0),h_1]+\phi^l([h_0,h_2],h_1,h_3)+\phi^l([h_1,h_3],h_0,h_2)=$ $\phi^l([h_0,h_1],h_2,h_3)+\phi^l([h_0,h_3],h_1,h_2)+\phi^l([h_1,h_2],h_0,h_3)+\phi^l([h_2,h_3],h_0,h_1)$ $+[\phi^l(h_1,h_2,h_3),h_0]+[\phi^l(h_3,h_0,h_1),h_2]$.

where $\phi^l(h_0, h_1, h_2) = m_H\big(0, \phi(t_H(h_0), t_H(h_1), t_H(h_2))\big)$

Morphisms can also be re-interpret in classical terms.

Stacky Lie algebroids

A morphism $(F, R) : (H, Q, \alpha) \to (H', Q, \alpha')$ between qLA-groupoids is a **Morita map** iff $F : H \to H'$ is VB-Morita as defined by Del Hoyo-Ortiz.

Two qLA-groupoids over $M \rightrightarrows M$, $(H_1, [\cdot,\cdot]_1, a_1, \phi_1)$ and $(H_2, [\cdot,\cdot]_2, a_2, \phi_2)$ are **Morita equivalent** if there exists $(H_3, [\cdot,\cdot]_3, a_3, \phi_3)$ and Morita maps

$$H_1 \xleftarrow{F_1, R_1} H_3 \xrightarrow{F_2, R_2} H_2$$

such that F_i on the base are the identity on M.

Definition 3

A **stacky Lie algebroid** over M is the Morita class of a qLA-groupoid over $M \rightrightarrows M$.

Part C

Main properties of Stacky Lie algebroids.

L_∞-algebroids

Sheng-Zhu

A n-**term** L_∞-**algebroid** is a non-positively graded vector bundle $A = \oplus_{i=0}^{n-1} A_{-i} \to M$ together with an anchor $\rho : A_0 \to TM$ and graded antisymmetric brackets $\{l_i\}$ of degree $2-i$ such that

$$\sum_{i+j=r+1} \sum_{\sigma \in Sh(i,j-1)} K sgn(\sigma) l_j(l_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), \cdots, x_{\sigma(r)}) = 0$$

- For $n=1 \rightsquigarrow (A_0, \rho, [\cdot,\cdot])$ Lie algebroid,
- For $n=2 \rightsquigarrow (A_{-1} \xrightarrow{\partial} A_0, \rho, [\cdot,\cdot], \nabla, [\cdot,\cdot,\cdot])$.

Theorem Bonavolonta-Poncin There is an equivalence of categories between n-term L_∞-algebroids and degree n Q-manifolds.

The infinitesimal counterpart of Lie 2-groupoids are 2-term L_∞-algebroids Severa, Zhu, Li-Zhu, Severa-Siran...

Dold-Kan Correspondence

Theorem Gracia-Saz–Mehta: VB-groupoids $(H \rightrightarrows E; G \rightrightarrows M) \leftrightarrows$ 2-term representations up to homotopy of $G \rightrightarrows M$.

Observe that a 2-term representation up to homotopy of $M \rightrightarrows M$ is the same as a 2-term complex $C \xrightarrow{\partial} E$.

Theorem 1

There is a 1-1 correspondence between

- quasi LA-groupoids covering the unit groupoid.
- 2-term L_∞-algebroids.

Moreover, the quasi LA-groupoids are Morita equivalent iff the 2-term L_∞-algebroids are quasi-isomorphic.

Infinitesimal counterpart of Zhu's Guiding Theorem.

More equivalences

Corollary 1

There is a 1-1 correspondence between the following sets:

1. Stacky Lie algebroids.
2. Morita classes of quasi LA-groupoids over the unit groupoid.
3. Quasi-isomorphism classes of 2-term L_∞-algebroids.
4. Isomorphism classes of degree 2 Q-manifolds.
5. Isomorphism classes of VB-Courant algebroids.

1 $[E/(E \oplus C)] \Rightarrow M$,

$\downarrow\uparrow$ by definition,

2 $(E \oplus C \rightrightarrows E; M \rightrightarrows M)$,

$\downarrow\uparrow$ Theorem **1**,

3 $C \xrightarrow{\partial} E$

$\downarrow\uparrow$ Bonavolonta-Poncin,

4 $(M, \wedge E^* \otimes SymC^*)$

$\downarrow\uparrow$ Li-bland,

5 $(D \to E; C^* \to M)$

Quasi Poisson groupoids

Iglesias-Ponte–Laurent-Gengoux–Xu

A **quasi Poisson groupoid** $(G \rightrightarrows M, \pi, \psi)$ is a groupoid with $\pi \in \mathfrak{X}^2_{mul}(G)$ and $\psi \in \wedge^3 \Gamma A$ s.t.

$$[\pi, \pi] = \psi^l - \psi^r, \qquad [\pi, \psi^l] = 0.$$

Recently Bonechi–Ciccoli–Laurent-Gengoux–Xu introduced

$$\left\{ \begin{array}{c} +1 \textbf{ Shifted Poisson} \\ \textbf{differentiable stack} \end{array} \right\} \leftrightarrows \left\{ \begin{array}{c} \text{Morita class of a} \\ \text{quasi Poisson groupoid} \end{array} \right\}$$

Dual of a Stacky Lie algebroid

Recall that VB-groupoids have duals

$$\begin{array}{ccc} H & \rightrightarrows & E \\ \downarrow & & \downarrow \\ G & \rightrightarrows & M \end{array} \quad \leftrightarrows \quad \begin{array}{ccc} H^* & \rightrightarrows & C^* \\ \downarrow & & \downarrow \\ G & \rightrightarrows & M \end{array}$$

Theorem 2

The dual of a quasi LA-groupoid is a linear qPoisson VB-groupoid.

More concretely, let $((H \rightrightarrows E; G \rightrightarrows M), Q, \alpha)$ be a quasi LA-groupoid then $(H^* \rightrightarrows C^*, \pi, \psi)$ is a quasi Poisson groupoid and π and ψ are linear tensors.

Corollary 2

The dual of a Stacky Lie algebroid is a linear $+1$ shifted Poisson stack.

Part D

Work in progress:

SLie functor and integration of Courant algebroids.

SLie functor

We want to define the functor **SLie** : $\mathbb{SLG} \to \mathbb{SLA}$.

- Differentiation strategy:

$$\left\{\begin{array}{c}\text{Stacky Lie}\\ \text{algebroids}\end{array}\right\} \leftrightarrows \left\{\begin{array}{c}\text{qLA-groupoid}\\ \text{over unit}\end{array}\right\}/\text{Morita}$$

$$\uparrow \textbf{qLie}$$

$$\left\{\begin{array}{c}\text{Stacky Lie}\\ \text{groupoids}\end{array}\right\} \leftrightarrows \left\{\begin{array}{c}\text{quasi double groupoid}\\ \text{over unit}\end{array}\right\}/\text{Morita}$$

- Integration strategy:
  - No good idea right now!!
  - For strict ones, $\alpha = 0$, it is possible.
  - If $M = \{*\}$, integration of semistrict Lie 2-algebras, we believe it is in the literature.
  - Topological obstructions?

Integration of Courant algebroids

Theorem Severa-Roytenberg: There is a 1-1 correspondence between Courant algebroids and degree 2 symplectic Q-manifolds.

By Corollary 1 the isomorphism class of a degree 2 Q-manifold correspond to a stacky Lie algebroid (c.f. Bressler). Moreover this must carry a compatible $+2$ shifted symplectic structure.(Stacky bialgebroid?)

New viewpoint: Courant algebroids integrates to $+2$ shifted symplectic stacky Lie groupoids.

The $+2$ shifted symplectic stacky Lie groupoids can be thought of as a sort of quotient of the 2-groupoid integrating the Courant algebroid. Hence, more chances to be finite dimensional!!

The symplectic structure is $+2$ shifted because the tangent and cotangent complexes has length 3.

Thanks !!