Universal centralizers and Poisson transversals

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The universal centralizer

G semisimple algebraic group of adjoint type over $\mathbb{C}$

$\text{rank}(G) = l$

$\mathfrak{g} = \text{Lie } G$

The regular locus of $\mathfrak{g}$ is

$$\mathfrak{g}^{\mathrm{r}} = \{x \in \mathfrak{g} \mid \dim G^x = l\}.$$

- this is the regular locus of the KKS Poisson structure
- x regular semisimple $\rightsquigarrow$ G^x is a maximal torus
- x regular nilpotent $\rightsquigarrow$ G^x is a abelian group $\cong \mathbb{C}^l$

The universal centralizer

Let $\{e, h, f\} \subset \mathfrak{g}$ be a regular $\mathfrak{sl}_2$-triple.

Theorem (Kostant)

The principal slice

$$\mathcal{S} = f + \mathfrak{g}^e \subset \mathfrak{g}^{\mathrm{r}}$$

meets each regular G-orbit on $\mathfrak{g}$ exactly once, transversally.

Remark

$\mathcal{S}$ is a Poisson transversal for the KKS Poisson structure.

Definition

The universal centralizer of $\mathfrak{g}$ is

$$\begin{array}{c} \mathcal{Z} = \{(a, x) \in G \times \mathfrak{g} \mid x \in \mathcal{S}, a \in G^x\} \\ \downarrow \\ \mathcal{S}. \end{array}$$

The universal centralizer

$\mathcal{Z}$ is a smooth, symplectic variety:

$$\begin{array}{c} G \times G \circlearrowright T^*_G \cong G \times \mathfrak{g} \\ \downarrow \mu \\ \mathfrak{g} \times \mathfrak{g} \end{array}$$

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$$
\begin{array}{ccc}
G \times G \circlearrowright T^*_G \cong G \times \mathfrak{g} & & (a, x) \\
\downarrow \mu & & \downarrow \mu \\
\mathfrak{g} \times \mathfrak{g} & & (a \cdot x, x)
\end{array}
$$

- $\mu^{-1}(x, x) = G^x \quad \Rightarrow \quad \mathcal{Z} = \mu^{-1}(\mathcal{S}_\Delta)$.
- the image of μ is $\{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid x \in G \cdot y\}$

$$\Rightarrow \mathcal{Z} = \mu^{-1}(\mathcal{S}_\Delta) = \mu^{-1}(\mathcal{S} \times \mathcal{S})$$

is a Poisson transversal in T^*_G.

The universal centralizer

G has a canonical smooth compactification $\overline{G}$, called the wonderful compactification.

Plan

Compactify the centralizer fibers of $\mathcal{Z}$ in $\overline{G}$.

$$G \rightsquigarrow \overline{G}$$

$$T^*_G \rightsquigarrow T^*_{\overline{G},D}$$

Extend the symplectic structure on $\mathcal{Z}$ to a log-symplectic structure on its partial compactification.

The partial compactification of $\mathcal{Z}$

Let $\tilde{G}$ be the simply-connected cover of G,
V a regular irreducible $\tilde{G}$-representation.

Definition (DeConcini–Procesi)

$$\begin{array}{ccc} \tilde{G} & \longrightarrow & (\mathrm{End}\ V)\backslash\{0\} \\ \downarrow & & \downarrow \\ G & & \mathbb{P}(\mathrm{End}\ V). \end{array}$$

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$$\begin{array}{ccc} \tilde{G} & \longrightarrow & (\mathrm{End}\ V)\backslash\{0\} \\ \downarrow & & \downarrow \\ G & \overset{\chi}{\hookrightarrow} & \mathbb{P}(\mathrm{End}\ V). \end{array}$$

The wonderful compactification of G is $\overline{G} := \overline{\chi(G)}$.

- independent of V
- smooth projective $G \times G$-variety
- $D := \overline{G}\backslash G$ is a simple normal crossing divisor

The partial compactification of $\mathcal{Z}$

Example

Let $G = PGL_2 \quad \rightsquigarrow \quad \tilde{G} = SL_2, \quad V = \mathbb{C}^2$. Then

$$\chi(G) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}(M_{2\times 2}) \mid ad - bc \neq 0 \right\},$$

and $\overline{G} = \mathbb{P}(M_{2\times 2}) \cong \mathbb{P}^3$.

$$D = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}(M_{2\times 2}) \mid ad - bc = 0 \right\} \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Non-example

Let $G = PGL_n$ for $n \geqslant 3$. Then $V = \mathbb{C}^n$ is not a regular rep of $\tilde{G} = SL_n$, and

$$\overline{G} \ncong \mathbb{P}^{n^2-1}.$$

The partial compactification of $\mathcal{Z}$

$$D = D_1 \cup \ldots \cup D_l.$$

$G \times G$-orbits on $\overline{G}$ $\quad\longleftrightarrow\quad$ $J \subset \{1, \ldots, l\}$, in the sense that

$$\overline{\mathcal{O}_J} = \bigcap_{j \in J} D_j.$$

For each $J \subset \{1, \ldots, l\}$: parabolic subgroups P_J and P_J^-
common Levi $L_J := P_J \cap P_J^-$,
corresponding Lie algebras $\mathfrak{p}_J, \mathfrak{p}_J^-, \mathfrak{l}_J$.

$$\begin{array}{ccc} \overline{L_J/Z(L_J)} & \hookrightarrow & \overline{\mathcal{O}_J} \\ & & \downarrow \\ & & G/P_J \times G/P_J^- \end{array}$$

The partial compactification of $\mathcal{Z}$

The log-cotangent bundle $T^*_{\overline{G},D}$ of $\overline{G}$ fits into a short exact sequence

$$T^*_{\overline{G},D} \hookrightarrow \overline{G} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow T_{\overline{G},D}.$$

$\rightsquigarrow$ $T^*_{\overline{G},D}$ is a Lie algebroid over $\overline{G}$ with trivial anchor map.

The fibers of $T^*_{\overline{G},D}$ are subalgebras of $\mathfrak{g} \times \mathfrak{g}$:

for each $J \subset \{1, \ldots, l\}$, there is a basepoint $z_J \in \mathcal{O}_J$ such that

$$T^*_{\overline{G},D,z_J} = \mathfrak{p}_J \times_{\mathfrak{l}_J} \mathfrak{p}_J^-.$$

The partial compactification of $\mathcal{Z}$

Definition

$$\begin{array}{c} \overline{\mathcal{Z}} = \left\{(a,x) \in \overline{G} \times \mathfrak{g} \mid x \in \mathcal{S}, a \in \overline{G^x}\right\} \\ \downarrow \\ \mathcal{S}. \end{array}$$

- generic fiber is a smooth toric variety
- special fibers are singular

The partial compactification of $\mathcal{Z}$

$T^*_{\overline{G},D}$ has a natural log-symplectic Poisson structure, and

$$G \times G \circlearrowright T^*_{\overline{G},D} \xrightarrow{\overline{\mu}} \mathfrak{g} \times \mathfrak{g}.$$

- $\overline{\mu}$ is projection onto the fibers of $\overline{G} \times \mathfrak{g} \times \mathfrak{g}$.
- the image of $\overline{\mu}$ is $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$.

$\Rightarrow \overline{\mu}^{-1}(\mathcal{S}_\Delta) = \overline{\mu}^{-1}(\mathcal{S} \times \mathcal{S}) \subset T^*_{\overline{G},D}$ is a Poisson transversal.

Theorem (B.)

$$\overline{\mathcal{Z}} \cong \overline{\mu}^{-1}(\mathcal{S}_\Delta) \subset T^*_{\overline{G},D}$$

is a smooth, log-symplectic partial compactification of $\mathcal{Z}$.

The multiplicative analogue

Plan

Integrate this to a multiplicative picture:

$$\mathfrak{g} \rightsquigarrow \tilde{G}.$$

$G \circlearrowright \tilde{G}$ by conjugation

$\rightsquigarrow$ corresponding regular locus

$$\tilde{G}^{\mathrm{r}} = \{g \in \tilde{G} \mid \dim G^g = l\}.$$

Remark

This is the regular locus of the AKM quasi-Poisson structure on $\tilde{G}$, whose nondegenerate leaves are the conjugacy classes.

The multiplicative analogue

Theorem (Steinberg)

There is an l-dimensional affine subspace

$$\Sigma \subset \tilde{G}^{\mathrm{r}}$$

which meets each regular conjugacy class in $\tilde{G}$ exactly once, transversally.

Definition

The (multiplicative) universal centralizer of $\tilde{G}$ is

$$\begin{array}{c} \mathfrak{Z} = \left\{(a,g) \in G \times \tilde{G} \mid g \in \Sigma, a \in G^g\right\} \\ \downarrow \\ \Sigma. \end{array}$$

The multiplicative analogue

The double $\mathbb{D}_G := G \times \tilde{G}$ has a natural q-Poisson structure with group-valued moment map

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} \circlearrowright & \mathbb{D}_G \\ & \downarrow \mu \\ & \tilde{G} \times \tilde{G} \end{array}$$

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Proposition (Finkelberg-Tsymbaliuk)

$$\mathfrak{Z} = \mu^{-1}(\Sigma_\Delta) = \mu^{-1}(\Sigma \times \iota(\Sigma)) \subset \mathbb{D}_G$$

is a smooth, symplectic algebraic variety.

The multiplicative analogue

Recall the inclusions

$$\begin{array}{ccccc} T^*_G & \hookrightarrow & T^*_{\overline{G},D} & \hookrightarrow & \overline{G} \times \mathfrak{g} \times \mathfrak{g} \\ & \searrow & & \searrow & \downarrow \\ & & G & \hookrightarrow & \overline{G}. \end{array}$$

Proposition (B.)

$T^*_{\overline{G},D}$ integrates to a smooth subgroupoid

$$\begin{array}{ccc} \mathbb{D}_{\overline{G}} & \hookrightarrow & \overline{G} \times \tilde{G} \times \tilde{G} \\ & \searrow & \downdownarrows \\ & & \overline{G} \end{array}$$

whose source/target fiber at $z_I \in \overline{G}$ is $P_I \times_{L_I} P_I^-$.

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The multiplicative analogue

Definition

$$\overline{\mathfrak{Z}} := \left\{(a,g) \in \overline{G} \times \tilde{G} \mid g \in \Sigma, a \in \overline{G^g}\right\}.$$

$\mathbb{D}_{\overline{G}}$ has a Hamiltonian q-Poisson structure with moment map

$$G \times G \circlearrowright \mathbb{D}_{\overline{G}} \xrightarrow{\overline{\mu}} \tilde{G} \times \tilde{G}.$$

Theorem (B. in progress)

$$\overline{\mathfrak{Z}} \cong \overline{\mu}^{-1}(\Sigma_\Delta) = \overline{\mu}^{-1}(\Sigma \times \iota(\Sigma)) \subset \mathbb{D}_{\overline{G}}$$

is a smooth, log-symplectic partial compactification of $\mathfrak{Z}$.