Lie groupoids and Logarithmic connections

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Plan of talk

Study flat connections on principal bundles with logarithmic singularities, using tools from the theory of Lie groupoids.

Principal bundles

- $\pi : P \to X$ a principal G-bundle,
- G a connected complex reductive group,
- Main example: $G = GL(n, \mathbb{C})$. Principal bundles in this case are equivalent to vector bundles.

Connections

A connection on P is a bundle map

$$\nabla : TX \to TP/G$$

such that $d\pi \circ \nabla = id$.

- Locally $\nabla = d + A$, $A \in \Omega^1(X, \mathfrak{g})$,
- $At(P) = TP/G$ has the structure of a Lie algebroid, the Atiyah algebroid. A connection ∇ is flat if ∇ is a Lie algebroid morphism.

Logarithmic singularities

- $D \subset X$ complex codimension 1 submanifold.
- $T_X(-\log D)$: Lie algebroid of vector fields on X which are tangent to D.
- A flat connection with logarithmic singularities along D is a Lie algebroid homomorphism

$$\nabla : T_X(-\log D) \to At(P),$$

such that $\nabla \circ d\pi = \rho$.
- Locally $\nabla = d + A\frac{dz}{z} + Bdx$.

Lie theoretic perspective

There are integrations of the various algebroids:

- $TX \rightsquigarrow \Pi(X)$ (ssc)
- $T_X(-\log D) \rightsquigarrow \Pi(X, D)$ (ssc)
- $At(P) \rightsquigarrow \mathcal{G}(P) = (P \times P)/G$.

Lie's Second theorem (Mackenzie-Xu, Moerdijk-Mrčun):

- $\nabla : TX \to At(P) \rightsquigarrow T : \Pi(X) \to \mathcal{G}(P)$
- $\nabla : T_X(-\log D) \to At(P) \rightsquigarrow T : \Pi(X, D) \to \mathcal{G}(P)$

Lie theoretic perspective

Theorem

Let $\mathcal{G}$ be a source simply connected Lie groupoid, with Lie algebroid A. There is an equivalence of categories

$$\mathrm{Rep}(A, G) \cong \mathrm{Rep}(\mathcal{G}, G).$$

Therefore, we study the representation theory of $\Pi(X, D)$.

Outline

1. Local theory : $\mathrm{Rep}(\Pi(\mathbb{A}, 0), G)$
2. Global theory : $\mathrm{Rep}(\Pi(X, D), G)$

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Local theory: ODE with Fuchsian singularity

We study differential equations on $\mathbb{A}$ of the form

$$z\frac{ds}{dz} = A(z)s,$$

where $A : \mathbb{A} \to \mathfrak{g}$, and $s : \mathbb{A} \to G$ is a fundamental solution.

Normal forms and classification results due to Levelt, Turrittin, Babbitt and Varadarajan, Kleptsyn and Rabinovich, Boalch, etc.

Normal form and classification

$$z\frac{ds}{dz} = A(z)s,$$

- Try to simplify by finding $s = g^{-1}t$, for $g : \mathbb{A} \to G$ such that

$$z\frac{dt}{dz} = A(0)t.$$

- Solution: $s(z) = g^{-1}z^{A(0)}$.
- Action of gauge transformation: $g * A = gAg^{-1} + zg'g^{-1}$.

Normal form and classification

$$z\frac{ds}{dz} = A(z)s,$$

- Want to find g such that:

$$gAg^{-1} + zg'g^{-1} = A(0).$$

- Solve order by order in z: $A = \sum_{i=0}^{\infty} z^i A_i$.
- At stage k, use $g_k = \exp(z^k X_k)$. Then

$$g_k * A = A_0 + z^k(A_k + [X_k, A_0] + kX_k) + O(z^{k+1}).$$

Let $X_k = (ad_{A_0} - k)^{-1}(A_k)$.

- Then $g = \Pi_i g_i$ solves the problem.

Resonance

If two eigenvalues of A_0 differ by a non-zero integer k, then $(ad_{A_0} - k)(X_k) = A_k$ may not admit a solution. The best we can hope for is the Levelt normal form

$$A(z) = S + \sum_{i \geq 0} z^i N_i,$$

where S semisimple, N_i nilpotent, and $[S, N_i] = iN_i$.

Existing classifications

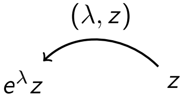
1. Classification in terms of analytic equivalence of Levelt normal form (Babbitt and Varadarajan, Kleptsyn and Rabinovich).
2. Classification in terms of monodromy operator and compatible Levelt filtration/ parabolic subgroup (Boalch).

These are difficult to make functorial because they use non-canonical normal forms.

Lie theoretic approach

Logarithmic connections on $\mathbb{A}$ are equivalent to representations of

$$\Pi(\mathbb{A}, 0) \cong \mathbb{C} \ltimes \mathbb{A} \rightrightarrows \mathbb{A}.$$



We study the category $\mathrm{Rep}(\mathbb{C} \ltimes \mathbb{A}, G)$ whose objects consist of a principal G-bundles $P \to \mathbb{A}$, and homomorphisms $\Phi : \mathbb{C} \ltimes \mathbb{A} \to \mathcal{G}(P)$.

Monodromy

- The monodromy of a representation (P, Φ) is $M(z) = \Phi(2\pi i, z)$.
- M is an automorphism of Φ.

Residue

- Groupoid homomorphism

$$\iota : \mathbb{C} \to \mathbb{C} \ltimes \mathbb{A}, \qquad \lambda \mapsto (\lambda, 0).$$

- Pullback functor $\iota^* : \mathrm{Rep}(\mathbb{C} \ltimes \mathbb{A}, G) \to \mathrm{Rep}(\mathbb{C}, G)$.
- $\iota^*(\Phi)(\lambda) = \exp(\lambda R)$, for $R \in \mathfrak{aut}_G(P_0)$, the residue of Φ.

Trivial representations

- Groupoid homomorphism

$$p : \mathbb{C} \ltimes \mathbb{A} \to \mathbb{C}, \qquad (\lambda, z) \mapsto \lambda.$$

- Pullback functor $p^* : \mathrm{Rep}(\mathbb{C}, G) \to \mathrm{Rep}(\mathbb{C} \ltimes \mathbb{A}, G)$.
- Representations in the image of p^* are trivial.

Linear approximation

$$L = p^* \circ \iota^* : \mathrm{Rep}(\mathbb{C} \ltimes \mathbb{A}, G) \to \mathrm{Rep}(\mathbb{C} \ltimes \mathbb{A}, G).$$

This functor takes an arbitrary representation and outputs the trivial representation determined by its residue.

Linearization

Definition

A linearization of a representation is an isomorphism

$$T : (P_0 \times \mathbb{A}, L(\Phi)) \to (P, \Phi).$$

The linearization is strict if $\iota^*(T) = id$.

- Can be thought of as a regularized parallel transport

$$T(1) : P_0 \to P_1.$$

- Linearizations encode the asymptotic nature of fundamental solutions at the singularity, and hence are closely related to the Levelt filtration.
- Linearizations do not always exist because of resonance.

Recall: Jordan Chevalley decomposition

An arbitrary element $g \in G$ has a unique decomposition of the form

$$g = su,$$

where s is semisimple, u is unipotent ($(u-1)^k = 0$), and $su = us$.

Linearization

Lemma

A representation Φ is linearizable if it has semisimple monodromy.

Proof. Recall the Levelt normal form for the associated differential equation:

$$z\frac{ds}{dz} = As, \qquad A(z) = S + \sum_{i\geq 0} z^i N_i,$$

where S semisimple, N_i nilpotent, and $[S, N_i] = iN_i$. Monodromy is given by

$$M(1) = \exp(2\pi i S)\exp(2\pi i N),$$

where $N = \sum_{i\geq 0} N_i$. Since M is semisimple, $N = 0$. ■

Recall: Groupoid 1-cocycles

- A 1-cocycle for $\mathbb{C} \ltimes \mathbb{A}$, valued in a representation (P, Φ), is a section σ of $t^* Aut_G(P)$ over $\mathbb{C} \ltimes \mathbb{A}$, which satisfies the following cocycle condition

$$\sigma(\mu, e^{\lambda} z)\Phi(\mu, e^{\lambda} z)\sigma(\lambda, z) = \sigma(\mu + \lambda, z)\Phi(\mu, e^{\lambda} z),$$

  for all $(\mu, \lambda, z) \in \mathbb{C} \times \mathbb{C} \times \mathbb{A}$.
- Given a representation Φ, and a cocycle σ, then $\sigma \circ \Phi$ is a new representation.

Untwisting cocycle

Theorem

Let (P, Φ) be a representation, and let U denote the unipotent part of its monodromy. Then the following defines a unipotent groupoid 1-cocycle

$$\sigma_\Phi(\lambda, z) = \exp(\frac{-\lambda}{2\pi i} log(U(e^\lambda z))).$$

The deformed representation

$$\Phi_s := \sigma_\Phi \circ \Phi,$$

has semisimple monodromy.

This defines a functorial Jordan Chevalley decomposition for representations.

Another look at resonance

Given a representation (P, Φ), the semisimple part Φ_s admits linearizations.

- The space of linearizations $\nu(\Phi_s)$ is a right torsor for $Aut(L(\Phi_s))$.
- The space of strict linearizations $\nu_0(\Phi_s)$ is a right torsor for $Aut_0(L(\Phi_s))$, the subgroup of automorphisms which are the identity above $0 \in \mathbb{A}$.
- $Aut_0(L(\Phi_s))$ is non-trivial if and only if the representation is resonant.

Another look at resonance

There is a split short exact sequence

$$1 \to Aut_0(L(\Phi_s)) \to Aut(L(\Phi_s)) \to Aut(\iota^*\Phi_s) \to 1.$$

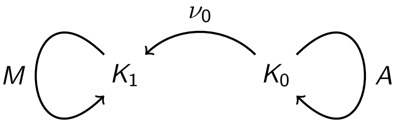
The splitting of this sequence is given by p^*.

Another look at resonance

- A linearization of Φ is equivalent to a linearization of Φ_s which takes U to $\iota^*(U)$.
- Choose an arbitrary linearization of Φ_s, which allows us to view $U \in Aut(L(\Phi_s))$. Then we are looking for an element of $Aut(L(\Phi_s))$ which conjugates U to $\iota^*(U)$.

Classification

Define a category $F(\mathbb{C}, G)$, whose objects are (M, K_1, ν_0, K_0, A)



1. K_0 and K_1 are right G-torsors,
2. $A = S + N \in \mathfrak{aut}_G(K_0)$,
3. $\nu_0 \subset \mathrm{Hom}_G(K_0, K_1)$, a right $\mathrm{Aut}_0(e^{\lambda S})$-torsor,
4. $M \in \mathrm{Aut}_G(K_1)$, which stabilizes $\nu_0 * \mathrm{Aut}(e^{\lambda S})$

such that $\pi(M) = \exp(2\pi i A)$.

Classification

Theorem

There is an equivalence of categories

$$\mathcal{L} : \mathrm{Rep}(\mathbb{C} \ltimes \mathbb{A}, G) \to F(\mathbb{C}, G),$$
$$(P, \Phi) \mapsto (M(1), P_1, \nu_0(\Phi_s), P_0, Res(\Phi)).$$

This functor has an explicit inverse $\mathcal{R} : F(\mathbb{C}, G) \to \mathrm{Rep}(\mathbb{C} \ltimes \mathbb{A}, G)$.

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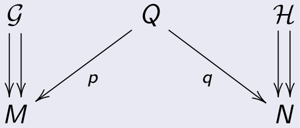
Global Theory: Representations of $\Pi(X, D)$

- We study the category of flat connections on X with logarithmic singularities on $D \subset X$ via the representations of $\Pi(X, D)$.
- Existing results due to Deligne, Simpson, Boalch, Ogus.
- Idea: Use Morita equivalence to reduce the representation theory of $\Pi(X, D)$ to the representation theory of $\mathbb{C} \ltimes \mathbb{A}$ and $\pi_1(X \setminus D)$.

Morita equivalence

Definition

A *Morita equivalence* between Lie groupoids $\mathcal{G} \rightrightarrows \mathcal{M}$ and $\mathcal{H} \rightrightarrows \mathcal{N}$ is a bi-principal $(\mathcal{G}, \mathcal{H})$ bi-bundle.



Morita equivalence

Definition

A Morita equivalence Q between $\mathcal{G}$ and $\mathcal{H}$ induces an equivalence of categories

$$\mathrm{Rep}(\mathcal{G}, G) \cong \mathrm{Rep}(\mathcal{H}, G)$$

Morita equivalence

A useful method for constructing Morita equivalences is the following result.

Criterion for Morita equivalent subgroupoid

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid $A \to M$, and $N \subseteq M$ an embedded submanifold. If N intersects every orbit of $\mathcal{G}$ and is transverse to A, then $\mathcal{G}|_N$ is a Lie subgroupoid of $\mathcal{G}$, which is Morita equivalent to $\mathcal{G}$.

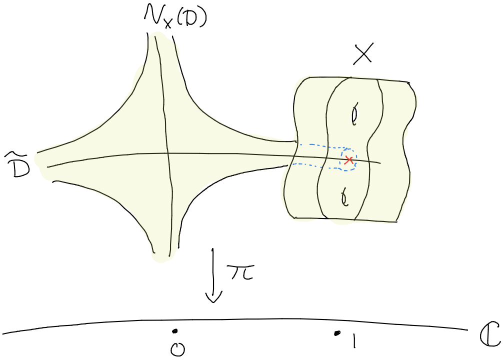
Deformation space

Construct a larger space

$$\pi : Z = \mathcal{D} \cup (X \times B(1, r)) \to \mathbb{C}$$

where $\mathcal{D}$ is the deformation to the normal cone of D in a tubular neighbourhood $D \subset U \subset X$.

Deformation space Z



- There is a codimension 1 submanifold $\tilde{D} \subseteq Z$.
- $\pi^{-1}(1) = (X, D)$ and $\pi^{-1}(0) = (N_X(D), 0)$.

Constructing the Morita equivalence

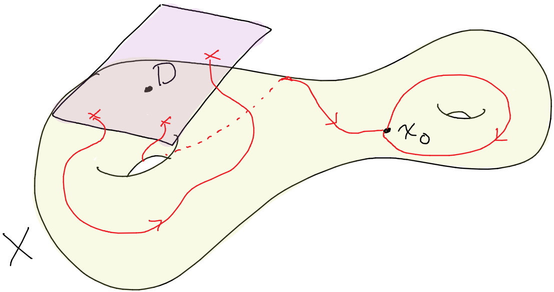
The groupoid $\Pi(Z, \tilde{D})$ has two Morita equivalent subgroupoids:

- $\Pi(Z, \tilde{D})|_X \cong \Pi(X, D)$
- $\mathcal{N} := \Pi(Z, \tilde{D})|_{N_X(D)|_d \cup \{x_0\}}$, determined by choice of $d \in D$ and $x_0 \in X \setminus D$.

Therefore

$$\mathrm{Rep}(\Pi(X, D), G) \cong \mathrm{Rep}(\Pi(Z, \tilde{D}), G) \cong \mathrm{Rep}(\mathcal{N}, G).$$

$\mathcal{N}$ = Groupoid of paths with tangential basepoints

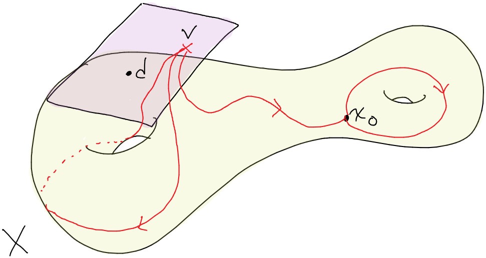


Subgroupoids of $\mathcal{N}$

Choose non-zero $v \in N_X(D)$.

- $\Pi(X, D)|_{\bar{v}} := \mathcal{N}|_{\{v, x_0\}}$
- $A(N_X(D)|_d) \ltimes N_X(D)|_d$, where

$$0 \to \mathbb{Z} \to \pi(N_X(D)^\times, v) \times \mathbb{C} \to A(N_X(D)|_d) \to 0.$$



Groupoid of paths with tangential basepoints

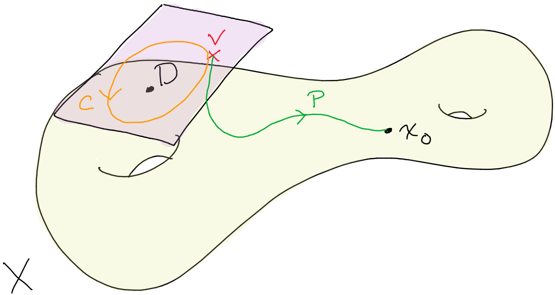
Theorem

Pushout of holomorphic Lie groupoids

$$\begin{array}{ccc} \pi(N_X(D)^{\times}, v) & \longrightarrow & \Pi(X \setminus D)_{\bar{v}} \\ \downarrow & & \downarrow \\ A(N_X(D)|_d) \ltimes N_X(D)|_d & \longrightarrow & \mathcal{N} \end{array}$$

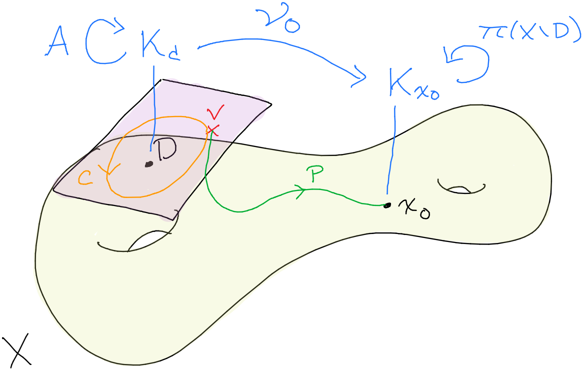
Classification

- Let $p : [0,1] \to X \setminus D$, such that $p(0) = d$, $p'(0) = v$, $p(1) = x_0$.
- Let c denote a loop in the fibre $N_X(D)^\times|_d$, and let $l = pcp^{-1} \in \pi_1(X \setminus D, x_0)$.

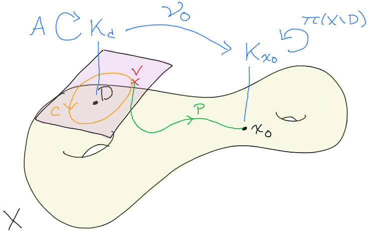


Classification

Define category $F(\pi(X \setminus D, x_0), G)$ with objects $(\Phi, K_{x_0}, \nu_0, K_d, A)$



$(\Phi, K_{x_0}, \nu_0, K_d, A)$



1. K_d and K_{x_0} are right G-torsors,
2. $A = S + N \in \mathfrak{aut}_G(K_d)$,
3. $\nu_0 \in \mathrm{Hom}_G(K_d, K_{x_0})$ is a right $\mathrm{Aut}_0(e^{\lambda S})$-torsor,
4. $\Phi : \pi_1(X \setminus D) \to \mathrm{Aut}_G(K_{x_0})$ is a homomorphism,

such that $\Phi(l)$ stabilizes $\nu_0 * \mathrm{Aut}(e^{\lambda S})$, and $\pi(\Phi(l)) = \exp(2\pi i A)$.

Classification

Theorem

There is an equivalence of categories

$$\mathrm{Rep}(T_X(-\log D), G) \cong F(\pi(X \setminus D, x_0), G).$$

Thank You