Poisson non-degeneracy of the Lie algebra $\mathfrak{so}(3,1)$

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Weinstein splitting theorem

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For $\mathrm{rank}(\pi(p)) = 2n$ there exists a local Poisson diffeomorphism

$$\phi : (M, \pi, p) \xrightarrow{\simeq} (\mathbb{R}^{2n}, \omega_{std}^{-1}, 0) \times (M', \pi_{M'}, p')$$

such that:

$$\pi_{M'}(p') = 0.$$

Example: linear Poisson = Lie algebra

If $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra $\rightsquigarrow$ $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ linear Poisson by:

$$\pi_{\mathfrak{g}}(\xi)(X, Y) := \xi([X, Y]) \quad \text{where } \xi \in \mathfrak{g}^*, X, Y \in \mathfrak{g}$$

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More explicitly: $\{x_i\}_{i=1}^n$ a basis for $\mathfrak{g}$ then:

$$[x_i, x_j] = c_{ij}^k x_k \quad \Rightarrow \quad \pi_{\mathfrak{g}} = c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

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Let $p \in M$ with $\pi_p = 0$ $\rightsquigarrow$ Lie algebra structure on $T_p^* M$:

$$[\mathrm{d}_p f, \mathrm{d}_p g] := \mathrm{d}_p \pi(\mathrm{d} f, \mathrm{d} g) \qquad \text{for } f, g \in C^\infty(M)$$

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Definition

- $(\mathfrak{g}_p = T_p^* M, [\cdot, \cdot])$ is the **isotropy Lie algebra** of π at p
- $(\mathfrak{g}_p^* = T_p M, \pi_{\mathfrak{g}_p})$ is the **linear approximation** of π at p

The linearization question: definition

Let $p \in M$ with $\pi(p) = 0$

Question

Is π linearizable at $p \in M$, i.e.

$$\exists\ \Phi : (M, \pi, p) \to (T_pM, \pi_{\mathfrak{g}_p}, 0)$$

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Definition

- A Lie algebra $\mathfrak{g}$ is called **Poisson non-degenerate**, if the answer is yes whenever $\mathfrak{g}_p \simeq \mathfrak{g}$
- otherwise $\mathfrak{g}$ is called **Poisson degenerate**

The linearization question: formal and analytic category

Theorem (Weinstein):

Any semisimple Lie algebra $\mathfrak{g}$ is formally non-degenerate.

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Theorem (Conn):

Any semisimple Lie algebra $\mathfrak{g}$ is analytically non-degenerate.

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Let $\mathfrak{g}$ be a **semisimple** Lie algebra $\rightsquigarrow$ Iwasawa decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

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For the linearization problem in the smooth category we have:

- if $\dim(\mathfrak{a}) = 0$ ($\mathfrak{g} = \mathfrak{k}$), Conn: $\mathfrak{g}$ is **Poisson non-degenerate** (Crainic & Fernandes: geometric proof)

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Conjecture (Dufour & Zung):

If $\dim(\mathfrak{a}) = 1$ and $\mathfrak{k}$ is semisimple, then $\mathfrak{g}$ is Poisson non-degenerate.

Let $\mathfrak{g} = \mathfrak{so}(3,1) \simeq \mathfrak{sl}_2(\mathbb{C})$ with Iwasawa decomposition:

$$\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}_2 \oplus \mathbb{R} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

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Theorem (Marcut & -):

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Strategy of the proof (Conn):

$$\left.\begin{array}{l} 2^{\text{nd}} \text{ Poisson cohomology} = 0 \\ + \text{ ``nice'' homotopy operators} \end{array}\right\} \xrightarrow{\text{Nash - Moser}} \text{Poisson non-degenerate}$$

Poisson cohomology: definition

The **Poisson cohomology** $H^{\bullet}(M, \pi)$ is obtained from $(\mathfrak{X}^{\bullet}(M), \mathrm{d}_\pi)$, where

$$\mathrm{d}_\pi := [\pi, \cdot] : \mathfrak{X}^{\bullet}(M) \to \mathfrak{X}^{\bullet+1}(M) \quad \text{with} \quad \mathrm{d}_\pi^2 = 0.$$

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- $H^2(M, \pi)$: infinitesimal deformations of π modulo deformations by diffeomorphisms

Poisson cohomology: linear Poisson

For $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ we have an isomorphism of complexes

$$(\mathfrak{X}^\bullet(\mathfrak{g}^*), \mathrm{d}_{\pi_{\mathfrak{g}}}) \simeq (\wedge^\bullet \mathfrak{g}^* \otimes C^\infty(\mathfrak{g}^*), \mathrm{d}_{EC}).$$

where d_{EC} is induced by the $\mathfrak{g}$-representation:

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The short exact sequence:

$$0 \to C_0^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}^*) \xrightarrow{j_0^\infty} \mathbb{R}[[\mathfrak{g}]] \to 0,$$

induces a long exact sequence in cohomology:

$$\ldots \xrightarrow{j_0^\infty} H_F^{\bullet-1}(\mathfrak{g}^*, \pi_\mathfrak{g}) \xrightarrow{\partial} H_0^\bullet(\mathfrak{g}^*, \pi_\mathfrak{g}) \to H^\bullet(\mathfrak{g}^*, \pi_\mathfrak{g}) \xrightarrow{j_0^\infty} H_F^\bullet(\mathfrak{g}^*, \pi_\mathfrak{g}) \xrightarrow{\partial} \ldots.$$

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Proposition

If $\mathfrak{g}$ is semisimple the l.e.s. becomes the short exact sequence

$$0 \to H_0^\bullet(\mathfrak{g}^*, \pi_{\mathfrak{g}}) \to H^\bullet(\mathfrak{g}^*, \pi_{\mathfrak{g}}) \xrightarrow{j_0^\infty} H_F^\bullet(\mathfrak{g}^*, \pi_{\mathfrak{g}}) \to 0.$$

Poisson cohomlogy: $(\mathfrak{so}(3,1)^*, \pi_{\mathfrak{so}(3,1)})$

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4. For calculating flat foliated cohomology, try to build a contraction to a "cohomological skeleton".

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2. There are homotopy operators

$$\mathfrak{X}^1_0(\overline{B}) \xleftarrow{h^1} \mathfrak{X}^2_0(\overline{B}) \xleftarrow{h^2} \mathfrak{X}^3_0(\overline{B})$$

satisfying for every $W \in \mathfrak{X}^2_0(\overline{B})$:

$$W = [\pi_{\mathfrak{so}(3,1)}, h^1(W)] + h^2([\pi_{\mathfrak{so}(3,1)}, W])$$

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and for all $n \in \mathbb{N}_0$:

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for some constants $d \in \mathbb{N}$, $0 < C$ and $i = 1, 2$.

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For $X \in \mathfrak{X}_0^j(\overline{B})$ and $k, n \in \mathbb{N}_0$:

$$\|X\|_{n,k} := \sup_{0 \leq |\alpha| \leq n} \sup_{x \in \overline{B}} \frac{1}{|x|^k} \left| \frac{\partial^{|\alpha|} X(x)}{\partial x^\alpha} \right|$$

Linearization

Let (M, π, p) be such that $\pi(p) = 0$ and $\mathfrak{g}_p \simeq \mathfrak{so}(3,1)$:

1. Use formal linearization (Weinstein) $\rightsquigarrow$ $\tilde{\pi} = \varphi^*(\pi)$ such that
$$\tilde{\pi}(p) = 0 \quad \text{and} \quad j_p^\infty \tilde{\pi} = \pi_{\mathfrak{so}(3,1)}$$
2. Apply Nash-Moser technique in the flat setting $\rightsquigarrow$ local diffeomorphism ϕ such that
$$\phi^*(\tilde{\pi}) = \pi_{\mathfrak{so}(3,1)}$$

Nash-Moser technique

- Newton-method: find zero of $f : \mathbb{R} \to \mathbb{R}$

$$|f(x_0)| < \epsilon \quad \Rightarrow \quad x_{l+1} = x_l - \frac{f(x_l)}{Df(x_l)}$$

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Nash-Moser in the proof:

$\pi_{\mathfrak{g}_p} - \pi \in \mathfrak{X}^2_0(M)$ and $\|\pi_{\mathfrak{g}_p} - \pi\|_n < \epsilon$ then

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$$X_l := S_l(h^1(\pi_{\mathfrak{g}_p} - \pi_l)) \quad \text{and} \quad \pi_{l+1} := \phi^*_{X_l}(\pi_l) \xrightarrow{l \to \infty} \pi_{\mathfrak{g}_p}$$

and $\phi := \Pi_{i=1}^{\infty} \phi_{X_i}$ defines a local diffeo. with $\phi^* \pi = \pi_{\mathfrak{g}_p}$

Thanks for your attention!