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Symplectic groupoids of elliptic Poisson manifolds

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Goal

Report on work in progress on:

- The geometry of elliptic Poisson manifolds;
- Construction of their associated adjoint groupoids, in particular their adjoint symplectic groupoid.

Outline

Outline

1. Elliptic Poisson manifolds
    - Poisson structures of divisor-type
    - Geometry of elliptic Poisson manifolds
    - Blowing-up elliptic Poisson structures
2. Background on symplectic groupoids
3. Groupoids for elliptic Poisson manifolds

Poisson structures of divisor-type

Definition

A **divisor** on X is a pair (U, σ) of a line bundle $U \to X$ and a section $\sigma \in \Gamma(U)$ whose zero set $Z_\sigma = \sigma^{-1}(0)$ is nowhere dense.

A divisor (U, σ) induces a divisor ideal $I_\sigma := \sigma(\Gamma(U^*))$.

Example

- **Log divisors** where σ has transverse zeroes along a hypersurface Z. Locally $I_Z = \langle z \rangle$;
- **Elliptic divisors** where σ vanishes along a codimension-two submanifold D where its normal Hessian $\mathrm{Hess}(\sigma) \in \Gamma(D; \mathrm{Sym}^2 N^* D \otimes U)$ is definite. Locally $I_{|D|} = \langle r^2 \rangle$.

Poisson structures of divisor-type

A Poisson manifold (X^{2n}, π) gives rise to a pair, namely $\mathrm{div}(\pi) := (\wedge^{2n} TX, \wedge^n \pi)$, with ideal $I_\pi \subseteq C^\infty(X)$.

Definition

A Poisson structure π on X is:

- **log-Poisson** if $\mathrm{div}(\pi)$ is a log divisor;
- **elliptic Poisson** if $\mathrm{div}(\pi)$ is an elliptic divisor.

Poisson structures of divisor-type

Definition

A Poisson structure π on X is:

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- **elliptic Poisson** if $\mathrm{div}(\pi)$ is an elliptic divisor.

The first of these: also *b*-Poisson, log-symplectic, *b*-symplectic (**Guillemin–Miranda–Pires** and several others).

The second are due to **Cavalcanti–Gualtieri** via stable GCS's.

Geometry of elliptic Poisson manifolds

Let (X, π) be elliptic Poisson. Then π is nondegenerate on $X \backslash D$. The degeneracy locus D is Poisson with $\pi_D \in \mathrm{Poiss}(D)$.

To describe π near/on D, we use Lie algebroids.

Given π we get an elliptic ideal I_π. This defines the **elliptic tangent bundle** $\mathcal{A}_{|D|} = TX(-\log|D|) \to X$, with

$$\Gamma(\mathcal{A}_{|D|}) \cong \left\{ v \in \Gamma(TX) \,|\, \mathcal{L}_v I_{|D|} \subseteq I_{|D|} \right\}.$$

Locally $\Gamma(\mathcal{A}_{|D|}) = \langle r\partial_r, \partial_\theta, \partial_{x_i} \rangle$ with (r, θ) normal to D.

Geometry of elliptic Poisson manifolds

We can 'lift' the Poisson structure π to $\mathcal{A}_{|D|}$, giving $\pi' \in \Gamma(\wedge^2 \mathcal{A}_{|D|})$ with $[\pi', \pi'] = 0$ and $\rho_{\mathcal{A}_{|D|}}(\pi') = \pi$.

The structure π' is nondegenerate, hence we get an **elliptic symplectic form** $\omega \in \mathrm{Symp}(\mathcal{A}_{|D|})$ with $\omega = \pi'^{-1}$.

We can study π using ω.

Geometry of elliptic Poisson manifolds

The elliptic tangent bundle has several **residue maps**, which are cochain maps. Assume that D carries a **coorientation** $\mathfrak{o}_D$.

- The **elliptic residue** is $\mathrm{Res}_D \colon \Omega^\bullet(\mathcal{A}_{|D|}) \to \Omega^{\bullet-2}(D)$;
- The **radial residue** is $\mathrm{Res}_r \colon \Omega^\bullet(\mathcal{A}_{|D|}) \to \Omega^{\bullet-1}(\mathrm{At}(S^1 ND))$;
- The θ-**residue** is $\mathrm{Res}_\theta \colon \Omega^\bullet_0(\mathcal{A}_{|D|}) \to \Omega^{\bullet-1}(D)$, with $\Omega^\bullet_0(\mathcal{A}_{|D|}) := \ker(\mathrm{Res}_D)$.

Locally for $\omega = d\log r \wedge d\theta \wedge \alpha + d\log r \wedge \beta + d\theta \wedge \gamma + \eta$,

- $\mathrm{Res}_D(\omega) = i_D^*(\alpha)$;
- $\mathrm{Res}_r(\omega) = (d\theta \wedge \alpha + \beta)|_D$;
- $\mathrm{Res}_\theta(\omega) = i_D^*(\gamma)$.

Note that $\mathrm{Res}_r \colon \Omega^\bullet_0(\mathcal{A}_{|D|}) \to \Omega^{\bullet-1}(D)$.

Geometry of elliptic Poisson manifolds

We apply these residue maps to powers of our symplectic form ω.

In particular, $\lambda := \mathrm{Res}_D(\omega) \in \Omega^0(D; \mathbb{R})$ (locally constant).

We say that π is **zero** or **nonzero** elliptic Poisson depending on λ.

Remark

The case $\lambda = 0$ corresponds to stable GC geometry.

Geometry of elliptic Poisson manifolds

Using Moser methods we obtain **pointwise Darboux models**:

Proposition

Let $(X^{2n}, |D|, \pi)$ be elliptic Poisson. Then near $x \in D$ we have

- $\pi = r\partial_r \wedge \partial_{x_1} + \partial_\theta \wedge \partial_{x_2} + \pi_0$ *if* $\lambda = 0$;
- $\pi = \lambda r \partial_r \wedge \partial_\theta + \pi_0$ *if* $\lambda \neq 0$,

where $\lambda = \mathrm{Res}_D(\omega)(p) \in \mathbb{R}$, *and* π_0 *is nondegenerate.*

We see π_D has rank $2n - 4$ if $\lambda = 0$, and rank $2n - 2$ if $\lambda \neq 0$.

Geometry of elliptic Poisson manifolds

More globally, ω orients $\mathcal{A}_{|D|}$ and hence TX, so X is **oriented**.

We can apply Res_D not just to ω, but to its spinor line $\langle e^\omega \rangle$.

This gives:

- When $\lambda = 0$, locally $\mathrm{Res}_D(e^\omega) = \pm\alpha_1 \wedge \alpha_2 \wedge e^\beta$, where $(\alpha_1, \alpha_2) = (\mathrm{Res}_r, \mathrm{Res}_\theta)(\omega) \in \Omega^1_{\mathrm{cl}}(D)^2$.

  In particular (locally) $\mathrm{Res}_q(\omega^n) = \pm n(n-1)\alpha_1 \wedge \alpha_2 \wedge \beta^{n-2} \neq 0$.

  Here β is pointwise inverse to π_D, and is **regular of corank** 2. Note that β is not globally closed on D, but $\alpha_1 \wedge \alpha_2 \wedge d\beta = 0$.

Geometry of elliptic Poisson manifolds

- When $\lambda \neq 0$, we instead define

$$\omega_D := \frac{1}{2\lambda}\mathrm{Res}_D(\omega^2).$$

After some computing, we see that

$$\mathrm{Res}_D(e^{\omega}) = \lambda e^{\omega_D} \neq 0.$$

Thus ω_D is symplectic and inverse to π_D, and **nondegenerate**.

Geometry of elliptic Poisson manifolds

Thus:

- X is always oriented by π;

Given a coorientation $\mathfrak{o}_D$ for D:

- If $\lambda = 0$, then π_D is corank-2 regular (almost 2-cosymplectic);
- If $\lambda \neq 0$, then π_D is nondegenerate.

When there is no coorientation, things are more involved.

Geometry of elliptic Poisson manifolds

There is also a **semi-global normal form**.

- Moser methods provide uniqueness;
- Need to build the models (c.f. **Witte**).

Again assume a coorientation $\mathfrak{o}_D$ is given. Then (using $I_{|D|}$) there is a unique compatible complex structure on $p \colon ND \to D$.

Choose a complex connection ∇ on ND with curvature K. Then can construct elliptic one-forms $\rho, \Theta \in \Omega^1(ND; \mathcal{A}'_{|D|})$ with $d\rho = -\mathrm{Re}(p^* K)$ and $d\Theta = -\mathrm{Im}(p^* K)$.

Geometry of elliptic Poisson manifolds

Theorem (Witte)

Let $(X, D, \mathfrak{o}_D, \pi)$ be a cooriented elliptic Poisson manifold. Then given ∇ there exists a tubular neighbourhood φ of D on which:

- *If $\lambda = 0$, then $\omega \cong \rho \wedge p^*(\alpha_1) + \Theta \wedge p^*(\alpha_2) + p^*(\omega_D)$, where*
  - $(\alpha_1, \alpha_2) = (\mathrm{Res}_r(\omega), \mathrm{Res}_\theta(\omega))$;
  - $\omega_D = (\varphi^*\omega - \rho \wedge p^*(\alpha_1) + \Theta \wedge p^*(\alpha_2))|_D$.
- *If $\lambda \neq 0$, then $\omega \cong \lambda\rho \wedge \Theta + p^*(\omega_D)$ with ω_D from before.*

More can be said about choices et cetera.

Geometry of elliptic Poisson manifolds

Using this normal form, similar to the log-Poisson case:

Proposition

Let $(X, D, \mathfrak{o}_D, \pi)$ be a cooriented zero elliptic Poisson manifold. Then π can be perturbed slightly so that it is ***proper****, i.e. the induced corank-2 Poisson structure π_D on D has compact leaves.*

In this case $p_{(\alpha_1, \alpha_2)} \colon D \to T^2$ is a fibration.

Blowing-up elliptic Poisson structures

We can **blow-up** elliptic Poisson structures to be log-Poisson.
Given an elliptic pair $(X, I_{|D|})$, by doing real blow-up we get

$$p\colon (\widetilde{X}, Z) \to (X, D),$$

with $\widetilde{X} = \mathrm{Bl}_D(X)$ and $Z = S^1 ND = \partial \widetilde{X}$.

Proposition (c.f. **Cavalcanti–Gualtieri, Kirchhoff-Lukat**)

Let $(X, I_{|D|})$ be an elliptic pair with blow-up $p\colon (\widetilde{X}, Z) \to (X, D)$. Then p induces a Lie algebroid morphism $(\varphi, p)\colon \mathcal{A}_Z \to \mathcal{A}_{|D|}$ with $\varphi \equiv dp$ on sections, which is a fiberwise isomorphism. Moreover, we have $\mathrm{Res}_Z \circ \varphi^* = p^* \circ \mathrm{Res}_r$ on $\Omega^\bullet_0(X; \mathcal{A}_{|D|})$.

Here $\mathcal{A}_Z = T\widetilde{X}(-\log Z)$ is the log-tangent bundle.

Blowing-up elliptic Poisson structures

Next, for elliptic Poisson manifolds we get (c.f. **Polishchuk**):

Proposition

Let (X, D, π) be an elliptic Poisson manifold with blow-up $p\colon (\widetilde{X}, Z) \to (X, D)$. Then $(\widetilde{X}, Z)$ admits a unique log-Poisson structure $\widetilde{\pi}$ such that p is a Poisson map: $p_*(\widetilde{\pi}) = \pi$.

When $\lambda = 0$:
Note that if π is proper, then so will be $\widetilde{\pi}$, with $\alpha = \mathrm{Res}_Z(\widetilde{\omega})$ satisfying $\alpha = p|_D^*(\alpha_1)$ for $\alpha_1 = \mathrm{Res}_r(\omega)$, giving $p_\alpha\colon Z \to S^1$.

Blowing-up elliptic Poisson structures

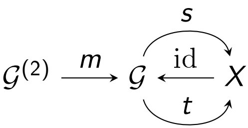
Summarizing, the blow-up procedure gives:

$$\begin{array}{ccc} (T\widetilde{X}(-\log Z), \widetilde{\omega}^{-1}) & \longrightarrow & (T\widetilde{X}, \widetilde{\pi}) \\ \varphi \downarrow & & \downarrow Tp \\ (TX(-\log |D|), \omega^{-1}) & \longrightarrow & (TX, \pi) \end{array}$$

Outline

Symplectic groupoids

We denote groupoids by $\mathcal{G} \rightrightarrows X$ and structure maps:



A **Poisson groupoid** carries a multiplicative Poisson structure $\pi_{\mathcal{G}}$, i.e. $\mathrm{Graph}(m)$ is coisotropic in $(\mathcal{G} \times \mathcal{G} \times \mathcal{G}, \pi_{\mathcal{G}} \oplus \pi_{\mathcal{G}} \oplus -\pi_{\mathcal{G}})$.

Symplectic groupoids

A Poisson groupoid is **symplectic** if $\pi_{\mathcal{G}}$ is nondegenerate.

A Lie groupoid $\mathcal{G} \rightrightarrows X$ has a Lie algebroid $\mathrm{Lie}(\mathcal{G}) \to X$, which it **integrates**. Not all Lie algebroids are integrable.

A Poisson manifold (X, π) has a Lie algebroid $T^*_\pi X \to X$. If it is integrable, its ssc integration is symplectic. Not all integrations need to be symplectic.

Adjoint groupoids

The ssc integration $\mathcal{G}(\mathcal{A})$ of a Lie algebroid $\mathcal{A}$, also called its **Weinstein groupoid**, can be seen as the largest integration.

The smallest integration is $\mathrm{Adj}(\mathcal{A})$, the **adjoint groupoid**, if it exists.

Given any integration $\mathcal{G}$ of $\mathcal{A}$, there are groupoid morphisms

$$\mathcal{G}(\mathcal{A}) \to \mathcal{G} \to \mathrm{Adj}(\mathcal{A}).$$

Adjoint groupoids

Debord showed the adjoint groupoid exists if $\mathcal{A}$ is almost-injective, i.e. if its anchor is injective on sections.

Androulidakis–Zambon define a Poisson manifold (X, π) to be **almost-regular** if $\mathcal{F}_\pi := \pi^\sharp(\Gamma(T^*X)) \subseteq \Gamma(TX)$ is projective.

This is precisely the case where its holonomy groupoid $\mathcal{H}(\mathcal{F}_\pi)$ is smooth, and thus when it admits a smooth adjoint integration. They also show it is a Poisson groupoid.

Adjoint groupoids

Any generically nondegenerate Poisson structure is almost-regular.

So, log- and elliptic Poisson manifolds have adjoint integrations.

Androulidakis–Zambon also show that if π is generically nondegenerate, then the adjoint groupoid for $T^*_\pi X$ is symplectic.

That means that all of its integrations will be symplectic.

Adjoint groupoids

Instead of $T^*_\pi X$, we will integrate $\mathcal{R}_\pi$ with $\Gamma(\mathcal{R}_\pi) \cong \mathcal{F}_\pi$ (they are isomorphic in our cases).

For (X, π) elliptic Poisson, there is a sequence

$$T^*_\pi X \to \mathcal{R}_\pi \to TX(-\log |D|) \to TX.$$

For $(\widetilde{X}, \widetilde{\pi})$ log-Poisson, we also have

$$T^*_{\widetilde{\pi}} \widetilde{X} \to \mathcal{R}_{\widetilde{\pi}} \to T\widetilde{X}(-\log Z) \to T\widetilde{X}.$$

Adjoint groupoids

We first recall how to construct groupoids for log-Poisson.

Example

The **pair groupoid** $\mathrm{Pair}(X) = X \times X \rightrightarrows X$, $\mathrm{Lie}(\mathrm{Pair}(X)) = TX$.

Example

Given a fibration $f\colon X \to X'$, there is the **relative pair groupoid**

$$\mathrm{Pair}_f(X) = X \times_f X = \{(x,y) \in \mathrm{Pair}(X) \,|\, f(x) = f(y)\} \subseteq \mathrm{Pair}(X).$$

with $\mathrm{Lie}(\mathrm{Pair}_f(X)) = \ker Tf$.

Elementary modification and blow-ups

Let $(\mathcal{A}, \mathcal{B}) \to (X, Z)$ be a Lie algebroid pair.

Definition (Gualtieri–Li)

The **lower elem. modif.** is $[\mathcal{A}{:}\mathcal{B}] \to X$ with

$$\Gamma([\mathcal{A}{:}\mathcal{B}]) \cong \{v \in \Gamma(\mathcal{A}) \mid v|_Z \in \Gamma(\mathcal{B})\}.$$

There is also **upper elem. modif.** $\{\mathcal{A}{:}\mathcal{B}\}$ using a Lie algebroid copair, and $[\mathcal{A}{:}\mathcal{B}]^* \cong \{\mathcal{A}^*{:}\mathcal{B}^*\}$ and $\{\mathcal{A}{:}\mathcal{B}\}^* \cong [\mathcal{A}^*{:}\mathcal{B}^*]$.

There are natural morphisms $[\mathcal{A}{:}\mathcal{B}] \to \mathcal{A}$ and $\mathcal{A} \to \{\mathcal{A}{:}\mathcal{B}\}$.

Elementary modification and blow-ups

There is a blow-up groupoid $[\mathcal{G}{:}\mathcal{H}]$ given a closed Lie groupoid pair.

Theorem (Gualtieri–Li)

Let $((\mathcal{G},\pi_{\mathcal{G}}),(\mathcal{H},\pi_{\mathcal{H}})) \rightrightarrows (X,Z)$ be a closed Poisson groupoid pair with $\mathrm{Lie}(\mathcal{G},\pi_{\mathcal{G}}) = (\mathcal{A},\mathcal{A}^*)$ and $\mathrm{Lie}(\mathcal{H},\pi_{\mathcal{H}}) = (\mathcal{B},\mathcal{B}^*)$. If the induced transverse Poisson structure on $N^*\mathcal{H}$ is degenerate, then $[\mathcal{G}{:}\mathcal{H}]$ inherits a multiplicative Poisson structure $\widetilde{\pi}_{\mathcal{G}}$ via blow-up and there is an induced surjective comorphism $\mathcal{A}^*|_Z \to \mathcal{B}^*$ such that

$$\mathrm{Lie}([\mathcal{G}{:}\mathcal{H}],\widetilde{\pi}_{\mathcal{G}}) = ([\mathcal{A}{:}\mathcal{B}],\{\mathcal{A}^*{:}\mathcal{B}^*\}).$$

In particular, in general $\mathrm{Lie}([\mathcal{G}{:}\mathcal{H}]) = [\mathcal{A}{:}\mathcal{B}]$, and $[\mathcal{G}{:}\mathcal{H}]^{\mathrm{c}}$ is adjoint if $\mathcal{G}$ was adjoint, where $\mathcal{G}^{\mathrm{c}} \subseteq \mathcal{G}$ is largest s-connected wide Lie subgroupoid by taking conn. components of identity in each s-fiber.

Elementary modification and blow-ups

Theorem (Gualtieri–Li)

Let $(\widetilde{X}, \widetilde{\pi})$ be proper log-Poisson with $p_\alpha\colon Z \to S^1$. Then:

- *The log pair groupoid $\mathrm{Pair}(\widetilde{X}, Z) := [\mathrm{Pair}(\widetilde{X}){:}\mathrm{Pair}^{\mathrm{c}}(Z)]^{\mathrm{c}}$ is the adjoint Poisson groupoid integrating $T\widetilde{X}(-\log Z)$, and carries a canonical log-Poisson structure $\widetilde{\sigma}$;*
- *The log-Poisson groupoid $\mathrm{Pair}_\pi(\widetilde{X}, Z) := [\mathrm{Pair}(\widetilde{X}){:}\mathrm{Pair}^{\mathrm{c}}_{p_\alpha}(Z)]$ is the adjoint symplectic groupoid (with $\widetilde{\tau}$) integrating the Poisson algebroid $T^*_{\widetilde{\pi}}\widetilde{X} \cong \mathcal{R}_{\widetilde{\pi}} = [T\widetilde{X}\colon \ker\langle\alpha\rangle]$.*

The following natural groupoid morphisms are Poisson:

$$(\mathrm{Pair}_{\widetilde{\pi}}(\widetilde{X}, Z), \widetilde{\tau}) \to (\mathrm{Pair}(\widetilde{X}, Z), \widetilde{\sigma}) \to (\mathrm{Pair}(\widetilde{X}), \widetilde{\pi} \oplus -\widetilde{\pi}).$$

Elementary modification and blow-ups

Summarizing, we get for $(\widetilde{X}, Z)$ proper log-Poisson with α:

$$\begin{array}{ccccc}
(\mathrm{Pair}_{\widetilde{\pi}}(\widetilde{X}, Z), \widetilde{\tau}) & \longrightarrow & (\mathrm{Pair}(\widetilde{X}, Z), \widetilde{\sigma}) & \longrightarrow & (\mathrm{Pair}(\widetilde{X}), \widetilde{\pi} \oplus -\widetilde{\pi}) \\
 & & \Big\downarrow \scriptstyle{\mathrm{Lie}} & & \\
[T\widetilde{X} \colon \ker\langle\alpha\rangle] & \longrightarrow & ([T\widetilde{X} \colon TZ], \widetilde{\omega}^{-1}) & \longrightarrow & (T\widetilde{X}, \widetilde{\pi})
\end{array}$$

Outline

1. Elliptic Poisson manifolds

2. Background on symplectic groupoids

3. Groupoids for elliptic Poisson manifolds
   - Quotients over varying base
   - The elliptic pair groupoid
   - Zero elliptic Poisson manifolds
   - Nonzero elliptic Poisson manifolds

Groupoids for elliptic Poisson manifolds

Consider the situation $p\colon (\widetilde{X}, Z, \widetilde{\pi}) \to (X, D, \pi)$.

The plan is to:

- Blow-up the elliptic Poisson structure (**done**);
- Construct groupoids over $(\widetilde{X}, Z)$ via Gualtieri–Li;
- Quotient these along p to groupoids over (X, D).

Quotients over varying base

Mackenzie defines the notion of a **(smooth) congruence**, as a pair $(S, R) \subseteq (\mathcal{G} \times \mathcal{G}, X \times X)$ satisfying certain conditions.

Theorem (Mackenzie)

There is a one-to-one correspondence between smooth congruences and Lie groupoid fibrations $(F, f)\colon (\mathcal{G}, X) \to (\mathcal{G}', X')$ *where* $\mathcal{G}' = \mathcal{G}/S$ *and* $X' = X/R$.

But, p is not a surjective submersion: congruences are not smooth.

However, $p|_{\widetilde{X}\setminus Z}$ and $p|_Z\colon Z \to D$ are.

Quotients over varying base

Can take a disjoint union of two **minimal** smooth congruences over saturated neighbourhoods!

Theorem

Let $p\colon (\widetilde{X}, Z) \to (X, D)$ *be the blow-up of an elliptic pair. Let* $\widetilde{\mathcal{G}} \rightrightarrows \widetilde{X}$ *be a Lie groupoid for which* Z *is* $\widetilde{\mathcal{G}}$*-invariant, and let* $(S_Z, R_Z) \subseteq (\mathrm{Pair}(\widetilde{\mathcal{G}}_Z), \mathrm{Pair}(Z))$ *be a minimal smooth congruence on the restriction* $\widetilde{\mathcal{G}}_Z \rightrightarrows Z$ *subordinate to* $p|_Z\colon Z \to D$*. Then:*

- *There exists a congruence* $(S, R) \subseteq (\mathrm{Pair}(\widetilde{\mathcal{G}}), \mathrm{Pair}(\widetilde{X}))$ *on* $\widetilde{\mathcal{G}}$ *subordinate to* p *which restricts to* (S_Z, R_Z) *over* Z*, and is smooth and minimal over* $\widetilde{X} \backslash Z$*;*
- *The quotient groupoid* $\mathcal{G} := \widetilde{\mathcal{G}}/S \rightrightarrows X$ *is smooth and has a Lie groupoid morphism* $(P, p)\colon (\widetilde{\mathcal{G}}, \widetilde{X}) \to (\mathcal{G}, X)$*, which is a Lie groupoid isomorphism over* $\widetilde{X} \backslash Z$*.*

The elliptic pair groupoid

Consider $p\colon (\widetilde{X}, Z, \widetilde{\pi}) \to (X, D, \pi)$ with coorientation $\mathfrak{o}_D$. Then $Z = S^1 ND$ becomes a principal $U(1)$-bundle $p|_D\colon Z \to D$.

Theorem

The minimal quotient $\mathrm{Pair}(X, |D|) := \mathrm{Pair}(\widetilde{X}, Z)/p \rightrightarrows X$ *is the adjoint groupoid for* $TX(-\log|D|)$.

$$\begin{array}{ccc} \mathrm{Pair}(\widetilde{X}, Z) & \longrightarrow & \mathrm{Pair}(\widetilde{X}) \\ \Phi \downarrow & & \downarrow P \\ \mathrm{Pair}(X, |D|) & \longrightarrow & \mathrm{Pair}(X) \end{array} \quad \overset{\mathrm{Lie}}{\rightsquigarrow} \quad \begin{array}{ccc} T\widetilde{X}(-\log Z) & \longrightarrow & T\widetilde{X} \\ \varphi \downarrow & & \downarrow Tp \\ TX(-\log|D|) & \longrightarrow & TX \end{array}$$

The elliptic pair groupoid

$$\begin{array}{ccc}
(\mathrm{Pair}(\widetilde{X}, Z), \widetilde{\sigma}) & \longrightarrow & (\mathrm{Pair}(\widetilde{X}), \widetilde{\pi} \oplus -\widetilde{\pi}) \\
\Phi \big\downarrow & & P \big\downarrow \\
(\mathrm{Pair}(X, |D|), \sigma) & \longrightarrow & (\mathrm{Pair}(X), \pi \oplus -\pi)
\end{array}$$

Need to also quotient the log-Poisson structure $\widetilde{\sigma}$ on $\mathrm{Pair}(\widetilde{X}, Z)$.

Zero elliptic Poisson manifolds

Consider $p\colon (\widetilde{X}, Z, \widetilde{\pi}) \to (X, D, \pi)$ with coorientation $\mathfrak{o}_D$, assume $\lambda = 0$, proper, with $(\alpha_1, \alpha_2) = (\mathrm{Res}_r, \mathrm{Res}_\theta)(\omega)$ and $\alpha = p^*(\alpha_1)$.

Theorem?

The adjoint symplectic groupoid integrating $T^*_\pi X \cong \mathcal{R}_\pi$ is given by

$$\mathrm{Pair}_\pi(X, |D|) := [\mathrm{Pair}(\widetilde{X}){:}\mathrm{Pair}_{p_{(\alpha_1,\alpha_2)}\circ p}(Z)]/p \rightrightarrows X.$$

Further, the natural groupoid morphisms $\mathrm{Pair}_\pi(X, |D|) \to \mathrm{Pair}(X, |D|) \to \mathrm{Pair}(X)$ *are Poisson.*

Zero elliptic Poisson manifolds

The aim is thus to get the following picture:

$$
\begin{array}{ccccccc}
([\mathrm{Pair}(\widetilde{X}):\mathrm{Pair}_{(\alpha,\beta)\circ p}(Z)],\Omega) & \longrightarrow & (\mathrm{Pair}_{\widetilde{\pi}}(\widetilde{X},Z),\widetilde{\tau}) & \longrightarrow & (\mathrm{Pair}(\widetilde{X},Z),\widetilde{\sigma}) & \longrightarrow & (\mathrm{Pair}(\widetilde{X}),\widetilde{\pi}\oplus-\widetilde{\pi}) \\
\Phi\downarrow & & & & \Phi\downarrow & & P\downarrow \\
(\mathrm{Pair}_{\pi}(X,|D|),\tau) & & \longrightarrow & & (\mathrm{Pair}(X,|D|),\sigma) & \longrightarrow & (\mathrm{Pair}(X),\pi\oplus-\pi) \\
 & & & & \mathrm{Lie}\;\rightsquigarrow & & \\
[T\widetilde{X}:\ker\langle p^*(\alpha),p^*(\beta)\rangle] & \longrightarrow & [T\widetilde{X}:\ker\langle p^*(\alpha)\rangle] & \longrightarrow & (T\widetilde{X}(-\log Z),\widetilde{\omega}^{-1}) & \longrightarrow & (T\widetilde{X},\widetilde{\pi}) \\
\varphi\downarrow & & & & \varphi\downarrow & & \downarrow \\
\mathcal{R}_{\pi} & & \longrightarrow & & (TX(-\log|D|),\omega^{-1}) & \longrightarrow & (TX,\pi)
\end{array}
$$

Nonzero elliptic Poisson manifolds

Consider $p\colon (\widetilde{X}, Z, \widetilde{\pi}) \to (X, D, \pi)$ with coorientation $\mathfrak{o}_D$, assume $\lambda = 0$ with nondegenerate Poisson structure π_D.

Theorem?

The adjoint symplectic groupoid integrating $T^*_\pi X \cong \mathcal{R}_\pi$ is given by

$$\mathrm{Pair}_\pi(X, |D|) := \boxed{???}/p \rightrightarrows X.$$

Further, the natural groupoid morphisms $\mathrm{Pair}_\pi(X, |D|) \to \mathrm{Pair}(X, |D|) \to \mathrm{Pair}(X)$ *are Poisson.*

Nonzero elliptic Poisson manifolds

Again we aim for the following picture:

$$
\begin{array}{ccccccc}
(\boxed{???}, \Omega) & \longrightarrow & (\mathrm{Pair}_{\widetilde{\pi}}(\widetilde{X}, Z), \widetilde{\tau}) & \longrightarrow & (\mathrm{Pair}(\widetilde{X}, Z), \widetilde{\sigma}) & \longrightarrow & (\mathrm{Pair}(\widetilde{X}), \widetilde{\pi} \oplus -\widetilde{\pi}) \\
\Phi \downarrow & & & & \Phi \downarrow & & P \downarrow \\
(\mathrm{Pair}_{\pi}(X, |D|), \tau) & & \longrightarrow & & (\mathrm{Pair}(X, |D|), \sigma) & \longrightarrow & (\mathrm{Pair}(X), \pi \oplus -\pi) \\
 & & & & \text{Lie} \wr & & \\
\boxed{???} & \longrightarrow & [T\widetilde{X} \colon \ker\langle\alpha\rangle] & \longrightarrow & (T\widetilde{X}(-\log Z), \widetilde{\omega}^{-1}) & \longrightarrow & (T\widetilde{X}, \widetilde{\pi}) \\
\varphi \downarrow & & & & \varphi \downarrow & & \downarrow \\
\mathcal{R}_{\pi} & & \longrightarrow & & (TX(-\log |D|), \omega^{-1}) & \longrightarrow & (TX, \pi)
\end{array}
$$

Questions

- Construction of adjoint symp. groupoid for nonzero elliptic.
- Quotient procedure for Poisson/pre-symplectic groupoids?
- Can we perform blow-up using elliptic ideals, not using $(\widetilde{X}, Z)$?

Thanks for your attention.