Canonical decomposition of rational maps

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Decomposition ideas in the study of complex objects

Thurston's "topology implies geometry" work in

- geometry of 3-manifolds;
- theory of surface homeomorphisms;
- dynamics of rational maps.

Example: Geometrization of 3-manifolds

In 2D: Every compact surface admits either

- spherical geometry (genus 0),
- Eucledian geometry (genus 1),
- hyperbolic geometry (genus ≥ 2).

Let M be a (compact, connected, orientable) topological 3-manifold.

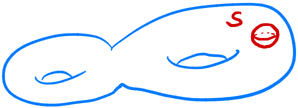
Question: Does M admits a geometric structure?
(M admits unique piecewise-linear and smooth structures [Moise'52])

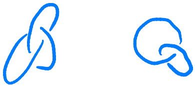
Thurston's vision on 3-manifolds [Thurston'80s - Perelman'00s]

Every **topological 3-manifold can be decomposed** naturally (along essential spheres and tori) into pieces so that each piece carries one of the eight standard geometric structures:

- spherical $\mathbb{S}^3$, Euclidean $\mathbb{R}^3$, hyperbolic $\mathbb{H}^3$;
- two products: $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$;
- three special geometries: Sol, Nil, $\widetilde{SL_2(\mathbb{R})}$.

An embedded sphere $S \subset M$ is non-essential if it bounds a solid 3-ball in M on at least one side, and is essential otherwise.



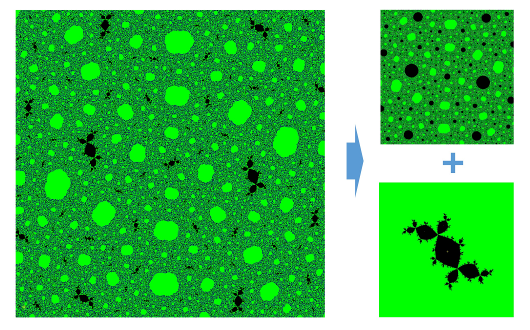


- [Kneser, Milnor, Jaco–Shalen, Johansson] canonical decompositions of M along essential spheres and tori.
- The manifolds with the 7 non-hyperbolic metrics all have some particular fibrations and are topologically classified [Seifert'33].

Decomposition theorem
(joint with Dima Dudko and Dierk Schleicher)

Every postcritically-finite rational map with non-empty Fatou set can be canonically decomposed into

- crochet maps (have "very thinly connected Julia sets") and
- Sierpiński carpet maps (have"very heavily connected Juila sets").



Dynamics of rational maps

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map.

- $\widehat{\mathbb{C}}$ is the Riemann sphere;
- $f^n = f \circ \cdots \circ f$ is the nth iterate of f.

Complex dynamics studies dynamical properties of rational maps under iteration.

$z_0 \in \widehat{\mathbb{C}}$ is a periodic point of period p if $f^p(z_0) = z_0$ and $p > 0$ is minimal.

The multiplier of z_0 is $\lambda := (f^p)'(z_0)$.

- If $|\lambda| < 1$, z_0 is called attracting;
- If $|\lambda| > 1$, z_0 is called repelling.

Example: $f(z) = -\frac{1}{3}(z^4 - 4z)$ has 5 fixed points: $0, 1, \omega = e^{2\pi i/3}, \omega^2 = e^{4\pi i/3}, \infty$.

- $f'(1) = f'(\omega) = f'(\omega^2) = f'(\infty) = 0$, thus $1, \omega, \omega^2, \infty$ are attracting:

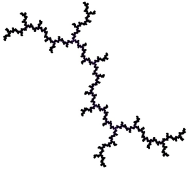
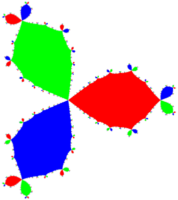
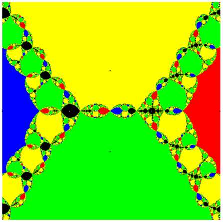
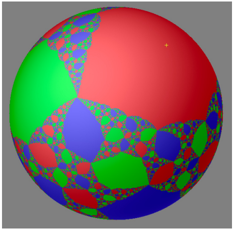
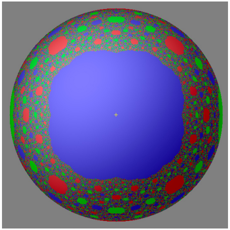
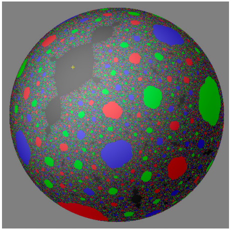
  if z is close to $z_0 \in \{1, \omega, \omega^2, \infty\}$, then $f^n(z) \xrightarrow[n\to\infty]{} z_0$.
- $f'(0) = \frac{4}{3}$, thus 0 is repelling.

Julia and Fatou sets

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map.

- The Julia set $\mathcal{J}_f$ is the closure of the set of repelling periodic points.
- The Fatou set $\mathcal{F}_f := \widehat{\mathbb{C}} \setminus \mathcal{J}_f$.
  A Fatou component is a connected component of $\mathcal{F}_f$.

Intuition: f behaves "regularly" on $\mathcal{F}_f$ and "chaotically" on $\mathcal{J}_f$.

Critical and postcritical set of a rational map

Each rational map $f: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a branched covering map, that is, f is

- continuous;
- surjective;
- locally $z \mapsto z^k$, $k \in \mathbb{N}$, after homeomorphic coordinate changes (k is called the local degree).

$c \in \widehat{\mathbb{C}}$ is a critical point if f is not locally injective at c.

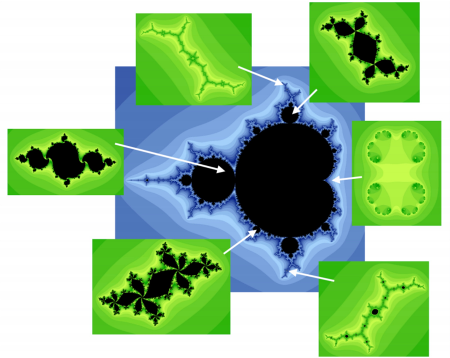
C_f – the set of all critical points of f.

$P_f := \bigcup_{n=1}^{\infty} f^n(C_f)$ – the postcritical set of f.

Global dynamics of f is controlled by the dynamics of C_f.

Figure: The Mandelbrot set $\mathcal{M}$ and the Julia sets of $f_c(z) = z^2 + c$

$\mathcal{M} = \{c \in \mathbb{C} : (f_c^n(0))_{n\in\mathbb{N}} \text{ is bounded}\} = \{c \in \mathbb{C} : \mathcal{J}_{f_c} \text{ is connected}\}$



Picture by S. Koch

f is postcritically-finite (pcf) if $\#P_f < \infty$, i.e., each critical point has finite orbit.

- pcf maps are rather special (it is a countable family),
- BUT! they are structurally very important.

Conjecture (McMullen)

The set of rational maps that are quasiconformally conjugate to pcf maps in a neighborhood of their Julia set is dense in the space of rational maps.

Julia and Fatou set for pcf rational maps

For a pcf rational map:

- $\mathcal{J}_f$ is a compact, connected, locally connected set in $\widehat{\mathbb{C}}$.
- $\mathcal{F}_f = \left\{ z \in \widehat{\mathbb{C}} : \{f^n(z)\}_{n \in \mathbb{N}} \text{ converges to a periodic critical cycle} \right\}$.
- Each Fatou component Ω is simply connected, $\partial\Omega$ is locally connected.

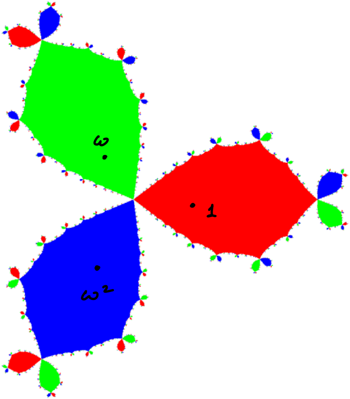
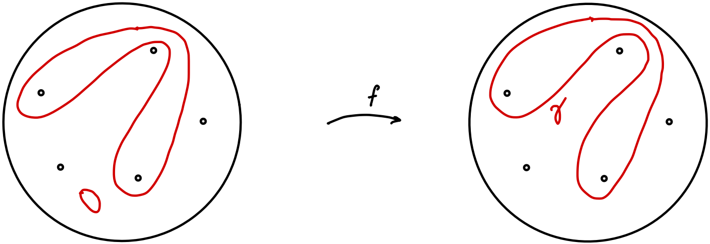


Figure: The Fatou and Julia set of $f(z) = -\frac{1}{3}(z^4 - 4z)$.

Decomposition scissors – Invariant multicurves

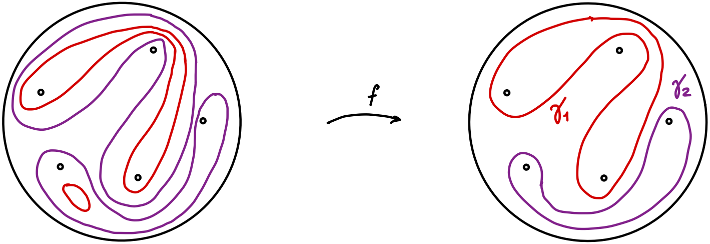
Let $f\colon S^2 \to S^2$ be a pcf branched covering map.

- $\mathscr{C}(f)$ – the set of all simple closed curves in $S^2 \setminus P_f$.



- $\gamma \in \mathscr{C}(f)$ is essential if each of the two connected components of $S^2 \setminus \gamma$ contains at least two points from P_f, and is peripheral otherwise.

- A multicurve is a non-empty finite family $\Gamma \subset \mathscr{C}(f)$ of essential curves that are pairwise disjoint and pairwise non-isotopic rel. P_f.



- A multicurve Γ is f-invariant if:

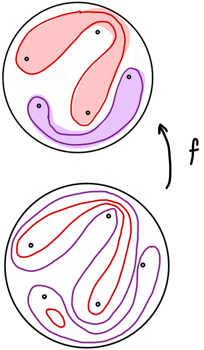
(i) $f^{-1}(\Gamma) \subset \Gamma$: each essential component of $f^{-1}(\Gamma)$ is homotopic to a curve in Γ.

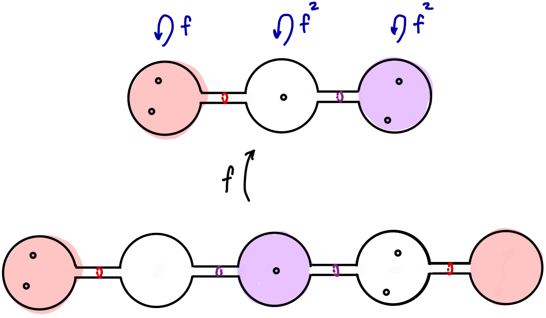
(ii) $\Gamma \subset f^{-1}(\Gamma)$: each curve in Γ is homotopic to a component of $f^{-1}(\Gamma)$.

Decomposition theory [Pilgrim]

Let $f\colon S^2 \to S^2$ be a pcf branched covering map with an invariant multicurve Γ.

A small sphere $\widehat{S^2}$ is a connected component of $S^2 \setminus \Gamma$, which we view as a finitely punctured sphere.





For a periodic (up to isotopy rel. P_f) small sphere $\widehat{S^2}$, the *first return map* $f^k\colon \widehat{S^2} \to \widehat{S^2}$ of f to $\widehat{S^2}$ is called a small map.

Decomposition results

Theorem (Thurston, Pilgrim, Selinger)

Let $f\colon S^2 \to S^2$ be a pcf branched covering map. Then there is a canonical multicurve Γ_{Th} (possibly empty) such that each small map $f^k\colon \widehat{S}^2 \to \widehat{S}^2$ is either

- *a homeomorphism (elliptic type);*
- *a double cover of a torus endomorphism (parabolic type);*
- *a rational map (hyperbolic type).*

If $\Gamma_{\mathrm{Th}} \neq \varnothing$ and f has hyperbolic orbifold then Γ_{Th} is the canonical Thurston obstruction.

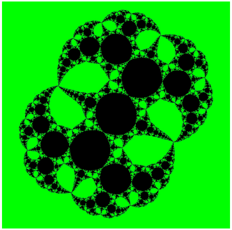
Theorem (Nielsen-Thurston)

Let S be a closed oriented surface and $f\colon S \to S$ be a homeomorphism. Then there is a canonical (i.e., maximal) multicurve Γ such that each small map $f^k\colon \widehat{S} \to \widehat{S}$ is either

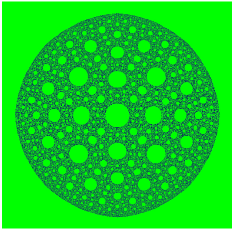
- *pseudo-Anosov;*
- *or periodic.*

Question: Is there a natural way to decompose pcf rational maps?

Idea: Use the structure of the Julia set! Namely, touching Fatou components

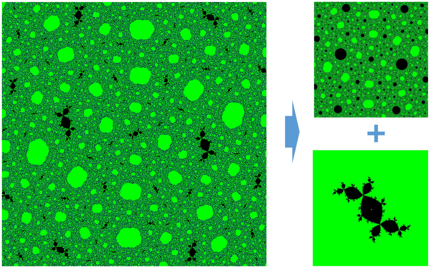


"many" touching Fatou components

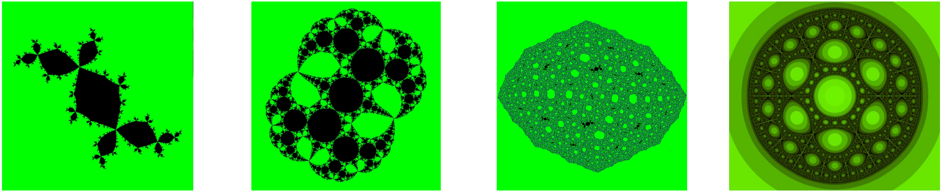


no touching Fatou components

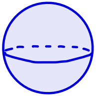
Extract maximal clusters of touching Fatou components



Cactoid quotient



$\mathcal{Q}(f)$:



Consider the equivalence relation $\sim_{\mathcal{F}}$ on $\widehat{\mathbb{C}}$ that collapses all Fatou components, i.e., $\sim_{\mathcal{F}}$ is the smallest closed equivalence relation on $\widehat{\mathbb{C}}$, s.t.,

$$\sim_{\mathcal{F}} \quad \supset \quad \{(x_1, x_2) : x_1, x_2 \in \overline{\Omega} \text{ for a Fatou component } \Omega\}.$$

The quotient $\mathcal{Q}(f) := \widehat{\mathbb{C}}/_{\sim_{\mathcal{F}}}$ is a sphere cactoid (a union of segments and spheres).

Decomposition theorem [Dudko-H.-Schleicher]

Let f be a pcf rational map.

- f is called a Sierpiński carpet map if $\mathcal{J}_f$ is homeomorphic to the standard Sierpiński carpet.
- f is called a crochet map (or a *Newton-like map*) if $\mathcal{Q}(f)$ is a single point.

  Equivalently, there is a finite f-invariant connected graph $\mathcal{G}$ with $P_f \subset \mathcal{G}$ such that $\mathcal{G} \cap \mathcal{J}_f$ is countable.

Theorem

Let f be a pcf rational map with $\mathcal{F}_f \neq \varnothing$. Then there exits a canonical multicurve Γ_{cro}, s.t., each small map in the decomposition of f along Γ_{cro} is either

- *a crochet map*
- *or a Sierpiński carpet map.*

True in a topological setup (Böttcher expanding Thurston maps). Allows to localize Thurston obstructions Γ_{Th}: if $\gamma \in \Gamma_{\text{Th}}$ then either $\gamma \in \Gamma_{\text{cro}}$ or γ is in a Sierpiński sphere.

Main ingredients of the proof

Crochet algorithm

(1) Compute maximal clusters of touching Fatou components.

(2) Decompose the map with respect to the boundary multicurve of the clusters.

(3) Iterate (1) and (2) for each small map until all small maps are crochet or Sierpiński.

(4) Glue small crochet maps that correspond to the same point in $\mathcal{Q}(f)$.

Main ingredients of the proof

Properties of the cactoid quotient $\mathcal{Q}(f)$

Consider $\widetilde{f}\colon \mathcal{Q}(f) \to \mathcal{Q}(f)$ and the semi-conjugacy $\pi_{\widetilde{f}}\colon \widehat{\mathbb{C}} \to \mathcal{Q}(f)$.

- Small crochet spheres project under $\pi_{\widetilde{f}}$ to (marked) points in $\mathcal{Q}(f)$ and small Sierpiński spheres project to spheres in $\mathcal{Q}(f)$.
- The quotient map $\widetilde{f}\colon \mathcal{Q}(f) \to \mathcal{Q}(f)$ is topologically expanding.
- Let $g\colon X \to X$ be another expanding quotient of f with the semi-conjugacy $\pi_g\colon \widehat{\mathbb{C}} \to X$. Then π_g factors through $\pi_{\widetilde{f}}$.

$$\begin{tikzcd}
 & \widehat{\mathbb{C}} \arrow[loop right, "f"] \arrow[dl, "\pi_{\widetilde{f}}"'] \arrow[d, "\pi_g"] \\
\mathcal{Q}(f) \arrow[loop left, "\widetilde{f}"] \arrow[r] & X \arrow[loop right, "g"]
\end{tikzcd}$$

Characterization of crochet maps [Dudko-H.-Schleicher]

Theorem

Let f be pcf rational map with $\mathcal{F}_f \neq \varnothing$. Then the following are equivalent.

(i) *f is a crochet map, that is, $\mathcal{Q}(f)$ is a single point;*

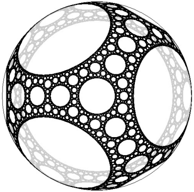
(ii) *there is a finite f-invariant connected graph $\mathcal{G}$ with $P_f \subset \mathcal{G}$ such that $\mathcal{G} \cap \mathcal{J}_f$ is countable.*

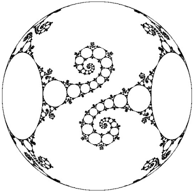
(iii) *there is a finite f-invariant connected graph $\mathcal{G}$ with $P_f \subset \mathcal{G}$ such that the topological entropy of $f|\mathcal{G}$ is 0;*

(iv) *$\mathcal{J}_f$ has countable separation property, that is, for each $x, y \in \mathcal{J}_f$ there is a countable subset $S \subset \mathcal{J}_f$ such that x and y belong to different connected components of $\mathcal{J}_f \setminus S$.*

Connections to geometric group theory

Sullivan's dictionary:
a framework relating dynamics of rational maps and Kleinian groups.





Limit spaces of Kleinian groups

Pictures by C. McMullen

- Similar objects, methods, proofs (Sullivan's no-wandering-domain theorems);
- The "rational dynamics analog" of Cannon's conjecture is established [Bonk-Meyer, Haïssinsky-Pilgrim].

Question

How can we measure "geometric complexity" of a fractal?

For example, consider Hausdorff dimension $\dim_{\mathrm{H}}$.

Conformal dimension of a metric space $\mathcal{X}$:

$$\mathrm{ConfDim}(\mathcal{X}) := \inf\{\dim_{\mathrm{H}}(\mathcal{Y}) : \text{metric spaces } \mathcal{Y} \text{ quasisymmetric to } \mathcal{X}\}$$

Similarly, one defines Ahlfors-regular conformal dimension ARConfDim.

(provide natural invariants for limit spaces of boundaries of Gromov hyperbolic groups)

For a pcf rational map f with the decomposing curve Γ_{cro}:

$$\mathrm{ARConfDim}(\mathcal{J}_f) \geq \max\left(\mathrm{ARConfDim}(\text{small Julia sets}), Q(\Gamma_{\mathrm{cro}})\right)$$

Theorem (Insung Park, based on a criterion by Pilgrim-D.Thurston)

A hyperbolic pcf rational map f is a crochet map if and only if $\mathrm{ARConfDim}(\mathcal{J}_f) = 1$.

Connections to geometric group theory

Iterated monodromy groups [Nekrashevych]

Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a pcf rational map and $t \in \widehat{\mathbb{C}} \setminus P_f$ be a basepoint.

$\pi_1(\widehat{\mathbb{C}} \setminus P_f, t) \curvearrowright f^{-1}(t)$ by the monodromy action

$\pi_1(\widehat{\mathbb{C}} \setminus P_f, t) \curvearrowright \bigcup_{n=0}^{\infty} f^{-n}(t)$ — iterated monodromy action

$\mathrm{IMG}(f) := \pi_1(\widehat{\mathbb{C}} \setminus P_f, t)/\ker$ — iterated monodromy group

- IMG's provide a useful algebraic tool and invariant in complex dynamics.
- $\mathrm{IMG}(f)$ is a self-similar group.

There is a natural identification $\bigcup_{n=0}^{\infty} f^{-n}(t) \longleftrightarrow \{\text{finite words in an alphabet } X\}$

$$\forall g \in \mathrm{IMG}(f), x_1 \in X \quad \exists h \in \mathrm{IMG}(f), \quad \text{s.t.} \quad g(x_1 x_2 \ldots x_n) = g(x_1) h(x_2 \ldots x_n)$$

IMG's frequently have "exotic" algebraic properties:

- $\mathrm{IMG}(z^2 + i)$ is of intermediate growth;
- $\mathrm{IMG}(z^2 - 1)$ is an amenable group of exponential growth.

Question

Are there connections between dynamical properties of rational maps and algebraic properties of their IMG's?

Amenability of IMG's

A pcf rational map f is a crochet map if and only if the $\mathrm{IMG}(f)$ is generated by an automaton of polynomial growth.
[Dudko-H.-Schleicher]

In this case, $\mathrm{IMG}(f)$ is amenable.
[Juschenko-Nekrashevych-de la Salle, Nekrashevych-Pilgrim-D. Thurston]

Question

Is there a natural decomposition for the limit spaces of contracting self-similar groups?

THANK YOU!