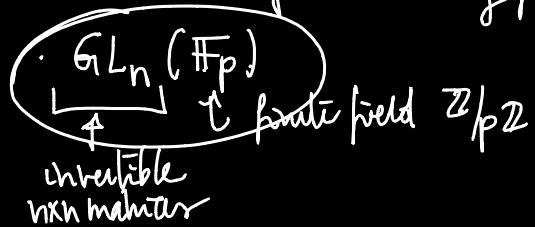


Endomorphisms of algebraic groups from the viewpoint of dynamical systems

joint work w./ Jakub Byszewski & Marc Houben
 'classics' construction of interesting finite groups



* algebraic group \approx "alg. var. G s.t.
 group operation $K \times K \rightarrow K$ & inverse $K \rightarrow K$
 are morphisms"

$$GL_2 \leftrightarrow \left\{ (a,b,c,d,t) \mid (ad-bc)t=1 \right\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow (a,b,c,d, \frac{1}{ad-bc}) \subseteq A^5$$

K a field \rightarrow $\mathcal{G}(K)$

* $\mathcal{G}_m =$ multiplicative group $\mathcal{G}_m(K) = K^*$

* $E =$ ell. curve (projective variety)

class 2 the prime number theorem (PNT)

$$\pi(x) = \#\text{primes} \leq x \dots$$

1712 Ramps $\lim_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} = 1$

Chebyshew $0 < \underline{c}_1 \leq \overline{c}_2 < \infty$

1896 proof : Hadamard - de la Vallée Poussin

better $\pi(x) = \text{Li}(x) + O(x^{\theta} \log x)$

$$\int_2^{+\infty} \frac{dt}{\log t} \quad \theta = \max \{ \Re(s) \mid \zeta(s) = 0 \}$$

$$\Re s = \frac{1}{2}$$

two computations

① $\# \text{SL}_n(\mathbb{F}_p) = \# \text{Aut}(\mathbb{F}_p^n)$
 $= (\frac{n}{p}-1)(p^n-p)(p^n-p^2) \cdots (p^n-p)$

② $P_l := \# \text{monic irreduc. polyn. over } \mathbb{F}_p \text{ of degree } l$

q=p $\pi(x) = \sum_{l \leq X} P_l \sim \frac{q}{q-1} \cdot \frac{q^X}{X} = \sum_{k \leq X^{\frac{1}{2}}} q^k + O\left(\frac{q^{\frac{1}{2}}}{X}\right)$

Reinterpret

① $\#\alpha_{L_n}(\mathbb{F}_q) = \#\text{fixed points}$ of
 q -Frobenius (" $x \rightarrow x^q$ ") acting on
 $\mathbb{A}^1(\mathbb{F}_q)$.

② $\overline{P_\ell} = \text{"prime" orbits of length } \ell \text{ for}$
 $q\text{-Frobenius acting on } \mathbb{A}^1(\mathbb{F}_q) = \mathbb{G}_{\mathrm{m}}(\mathbb{F}_q)$

General shift in dynamics

$S = \text{set} \quad S \xrightarrow{f} S \quad f^n = \underbrace{f \circ \dots \circ f}_n = S \xrightarrow{n} S$

- $f_n := \#\text{fixed points of } f^n \text{ in } S$
Assume: $f_n < \infty, \forall n$

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$$\zeta_f(z) = \exp \left(\sum_{n \geq 1} f_n \frac{z^n}{n} \right)$$

- orbits: $O_f(s) = \{ f^n(s) \}_{n \geq 0}$
of $s \in S$
periodic if $\exists n_0: f^{n_0}(s) = s$
(prime) orbit of length ℓ
 $O_f(s) = \{ s, f(s), \dots, f^\ell(s) = s \}$

$P_\ell := \#\text{prime orbits of length } \ell$

$$\pi_f(x) = \sum_{\ell \leq x} P_\ell$$

two results

$$\textcircled{1} \quad f_n = \sum_{\ell \mid n} \ell P_\ell$$

$$\textcircled{2} \quad \mathcal{I}_f(z) \in \mathbb{Q}(z)$$

$$f_n = \prod_{i=1}^n m_i \lambda_i^{n_i} \quad \lambda_i \in \mathbb{C}^* \\ m_i \in \mathbb{Z}$$

$$\mathcal{I}_f(z) = \prod_{i=1}^N (1 - \lambda_i z)^{-m_i}$$

Example

$$\textcircled{a} \quad \widehat{\mathbb{C}} \xrightarrow{f} \widehat{\mathbb{C}} \quad f \in \mathbb{C}(u), \text{ degree } \geq 2$$

Then $\mathcal{I}_f(z) \in \mathbb{Q}(z)$ [Hankkanen '94]

$$\textcircled{b} \quad X \text{ alg. var. } / \mathbb{F}_q \quad f_n = \#X(\mathbb{F}_{q^n})$$

$$f = \text{Frob}_q : X(\mathbb{F}_q) \rightarrow X(\overline{\mathbb{F}_q})$$

$\mathcal{I}_f(z)$ is Weil function $\in \mathbb{Q}(z)$ [Dwork, Artin-Weil]

C) $K = \text{alg closed field of char } p \geq 0$

$$f: K^* \xrightarrow{\cong} K^*: x \mapsto x^m$$

$$\mathbb{A}_m(K)$$

• $\text{char } K = 0, K = \mathbb{C}$

$$f_n = \#\text{sol}(x^m = x)$$

$$= \#\text{sol}(x^{m^n} = 1)$$

$$= m^n - 1$$

$$\text{If } f = \frac{1-z}{1-mz} \in \mathbb{Q}(z)$$

• $\text{char } K = p > 0$

$$f_n = \#\text{sol}(x^{m^n-1} = 1)$$

$$\begin{aligned} x^p &= 1 \\ x^{p-1} &= (x-1)^p \end{aligned}$$

$$f_n = \binom{m^n-1}{m^n-1}_p$$

Comparing
top
 $\downarrow p$

$$\binom{m^n-1}{m^n-1} = p^{r \cdot n}$$

$$\rightarrow \left[\begin{array}{l} \text{Thm. (A. Brumley 2012)} \\ \text{If } f \text{ is transcendental over } \mathbb{Q}(z) \\ \text{unless } p \mid m \end{array} \right]$$

Main theorem

- G (smooth connected) algebraic group over $\overline{\mathbb{F}_p}$
- $\sigma \in \text{End}(G)$ ($\sigma_n < \infty, \forall n$)
 - (A) $\sigma_n = d_n \cdot r_n^{-1} \cdot |n|_p^{s_n} \cdot p^{-t_n \ln_p^{-1}}$
 - d_n has rational zeros (\sim $\frac{\text{rat. for}}{\text{rat.}}$ $H^1_{\text{et}}(G, \overline{\mathbb{Q}_p})$)
 - $r_n \in \mathbb{Q}^*$, $s_n, t_n \in \mathbb{Z}_{\geq 0}$
 - periodization of period at.
 - (B) $\zeta_\sigma(z)$ is either rational $\in \mathbb{Q}(z)$ or transcendental over $\mathbb{Q}(z)$
 - (C) ("PNT") write $d_n = \sum m_i(\lambda_i^n)$
 - $\lambda = \max |\lambda_i|$
 - assume call σ hyperbolic if λ is the unique λ_i with $|\lambda| = |\lambda_i|$
 - $L_\sigma := \text{acc. pts. of } \left\{ \frac{x\pi_\sigma(x)}{x} \mid x \in \mathbb{Z}_0 \right\}$

C1 $\mathcal{L}_\sigma \subset \mathbb{R}_{>0}$ is compact (Chobyshev)

C2 \exists top-group \mathfrak{D}_σ [dilatation group]
 & map $i: \mathbb{Z} \rightarrow \mathfrak{D}_\sigma$
 s.t. $x_1, x_2, x_3, \dots \mapsto i(x_1), i(x_2), i(x_3), \dots$

$$\mathfrak{D}_\sigma = \left\{ \lim_{n \rightarrow \infty} \frac{x \pi_\sigma^n(x)}{n x} \mid \text{$i(x)$ converges} \right\}$$

C3 $\mathcal{L}_\sigma = \left\{ \frac{n}{n-1} \right\} \hookrightarrow$ a singleton

OR \mathcal{L}_σ uncountable
 ✓ if not hyperbolic, then contains an interval

~ of hyp. & \emptyset

$$\mathcal{L}_\sigma = \text{Cantor set} \cup \text{finite set}$$

$f_1 = \text{ab-var (ell-curves)}$ $D_\sigma = \max_i |\lambda_i(s)|$

$$D_\sigma(\lambda_i) = 0$$

- $t_n \equiv 0$

- rat. \mathcal{T} : σ acts nilpotently on $A[\mathbb{F}_p]$.

$\bullet \pi_\sigma(x) = \sum_{l \leq x} \frac{\lambda^l}{(re(l))^{se}} + O((\lambda^x))$