Infinite-dimensional Geometry : Theory and Applications

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FWF Grant I 5015-N

15th International Young Researchers Workshop on Geometry, Mechanics and Control

Outline

Lecture 1

Basics notions in infinite-dimensional geometry

Lecture 2

Inverse Function Theorems : Banach version and Nash-Moser version

Lecture 3

Some pathologies of infinite-dimensional geometry

1. Toys : Geometric structures
2. Traps of infinite-dimensional Geometry
3. Poisson Bracket not given by a Poisson tensor
4. Banach Poisson-Lie groups

What are the Toys we can play with?

Riemannian $\subset$ Symplectic $\subset$ Poisson Geometry

Riemannian metric = smoothly varying inner product on a manifold M

$$\begin{array}{cccc} g_x : & T_xM \times T_xM & \to & \mathbb{R} \\ & (U, V) & \mapsto & g_x(U, V) \end{array}$$

strong Riemannian metric = for every $x \in M$, $g_x : T_xM \to (T_xM)^*$ is an isomorphism

weak Riemannian metric = for every $x \in M$, $g_x : T_xM \to (T_xM)^*$ is just injective

Levi-Cevita connection may not exist for a weak Riemannian metric.

What are the Toys we can play with?

Riemannian $\subset$ **Symplectic** $\subset$ Poisson Geometry

Symplectic form = smoothly varying skew-symmetric bilinear form

$$\begin{array}{rccl} \omega_x : & T_xM \times T_xM & \to & \mathbb{R} \\ & (U, V) & \mapsto & \omega_x(U, V) \end{array}$$

with $dw = 0$ and $(T_xM)^{\perp_w} = \{0\}$

strong symplectic form = for every $x \in M$, $\omega_x : T_xM \to (T_xM)^*$ is an isomorphism
weak symplectic form = for every $x \in M$, $\omega_x : T_xM \to (T_xM)^*$ is just injective

Darboux Theorem does not hold for a weak symplectic form

What are the Toys we can play with?

Riemannian $\subset$ **Symplectic** $\subset$ Poisson Geometry

Hamiltonian Mechanics

(M, g) **strong Riemannian manifold**

- $$\begin{array}{cccc} \flat : & T_xM & \simeq & T_x^*M \\ & U & \mapsto & g_x(U, \cdot) \end{array} \qquad \flat^{-1} = \sharp$$
- **Kinetic energy = Hamiltonian**
  $$\begin{array}{cccc} H : & T^*M & \to & \mathbb{R} \\ & \eta_x & \mapsto & g_x(\eta_x^\sharp, \eta_x^\sharp) \end{array}$$

(T^*M, ω) **strong symplectic manifold**

- $\pi \; : T^*M \to M$
- $\omega = d\theta$
- $$\begin{array}{cccc} \theta_{(x,\eta)} : & T_{x,\eta}T^*M & \to & \mathbb{R} \\ & X & \mapsto & \eta(\pi_*(X)) \end{array} \qquad \textbf{Liouville 1-form}$$

geodesic flow = flow of Hamiltonian vector field $X_H : dH = \omega(X_H, \cdot)$

What are the Toys we can play with?

Riemannian $\subset$ **Symplectic** $\subset$ **Poisson Geometry**

Poisson bracket = family of bilinear maps
$\{\cdot,\cdot\}_U : \mathscr{C}^\infty(U) \times \mathscr{C}^\infty(U) \to \mathscr{C}^\infty(U)$, U open in M with

- skew-symmetry $\{f,g\}_U = -\{g,f\}_U$
- Jacobi identity $\{f,\{g,h\}_U\}_U + \{g,\{h,f\}_U\}_U + \{h,\{f,g\}_U\}_U = 0$
- Leibniz rule $\{f,gh\}_U = \{f,g\}_U h + g\{f,h\}_U$

A strong symplectic form defines a Poisson bracket by
$\{f,g\} = \omega(X_f, X_g)$ where $df = \omega(X_f,\cdot)$ and $dg = \omega(X_g,\cdot)$

A Poisson bracket may not be given by a bivector field

What are the Toys we can play with?

$$\left.\begin{array}{l}\text{Riemannian}\\ \text{Symplectic}\\ \text{Complex}\end{array}\right\} \subset \text{ Kähler} \subset \text{hyperkähler Geometry}$$

Complex structure = smoothly varying endomorphism J of the tangent space s.t. $J^2 = -1$.

Integrable complex structure : s. t. there exists an holomorphic atlas
Formally integrable complex structure : with Nijenhuis tensor $= 0$

Newlander-Nirenberg Theorem is not true in general :
formal integrability does not imply integrability.

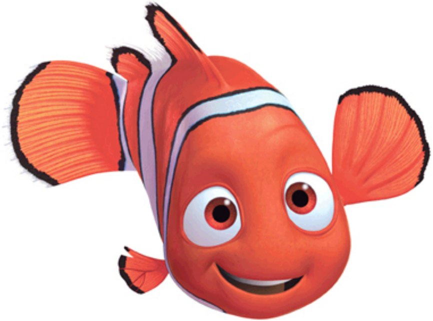
What are the traps of infinite-dimensional geometry?

In infinite-dimensional geometry, the golden rule is :
"Never believe anything you have not proved yourself!"

- The distance function associated to a Riemannian metric may by the zero function (Example by Michor and Mumford).
- Levi-Cevita connection may not exist for weak Riemannian metrics
- Hopf-Rinow Theorem does not hold in general : geodesic completeness $\neq$ metric completeness
- Darboux Theorem does not apply to weak symplectic forms
- A formally integrable complex structure does not imply the existence of a holomorphic atlas
- the tangent space differs from the space of derivations (even on a Hilbert space)
- a Poisson bracket may not be given by a bivector field (even on a Hilbert space)
- there are Lie algebras that can not be enlarged to Lie groups (Examples by Milnor or Neeb)

Poisson bracket not given by a Poisson tensor

Queer Poisson Bracket = Poisson bracket not given by a Poisson tensor



Poisson bracket not given by a Poisson tensor

$\mathscr{H}$ separable Hilbert space

Kinetic tangent vector $X \in T_x\mathscr{H}$ equivalence classes of curves $c(t)$, $c(0) = x$, where $c_1 \sim c_2$ if they have the same derivative at 0 in a chart.

Operational tangent vector $x \in \mathscr{H}$ is a linear map $D : C^\infty_x(\mathscr{H}) \to \mathbb{R}$ satisfying Leibniz rule :

$$D(fg)(x) = Df\ g(x) + f(x)\ Dg$$

Poisson bracket not given by a Poisson tensor

Ingredients :

- Riesz Theorem
- Hahn-Banach Theorem
- compact operators $\mathscr{K}(\mathscr{H}) \subsetneq \mathscr{B}(\mathscr{H})$ bounded operators

  $\Rightarrow \exists \ell \in \mathscr{B}(\mathscr{H})^*$ such that $\ell(\mathrm{id}) = 1$ and $\ell_{| \mathscr{K}(\mathscr{H})} = 0$.

Queer tangent vector [Kriegl-Michor]

Define $D_x : C^\infty_x(\mathscr{H}) \to \mathbb{R}$, $D_x(f) = \ell(d^2(f)(x))$, where the bilinear map $d^2(f)(x)$ is identified with an operator $A \in \mathscr{B}(\mathscr{H})$ by Riesz Theorem

$$d^2(f)(x)(X, Y) = \langle X, AY \rangle$$

Then D_x is an operational tangent vector at $x \in \mathscr{H}$ of order 2

Poisson bracket not given by a Poisson tensor

Queer tangent vector [Kriegl-Michor]

Let us show that $D_x(f) = \ell(d^2(f)(x))$ satisfies Leibniz rule :

$$D_x(fg)(x) = D_x f.g(x) + f(x).D_x g$$

The first and second derivatives of the product fg applied to $X, Y \in \mathscr{H}$ give

$$d(fg)(x)(X) = df(x)(X).g(x) + f(x).dg(x)(X)$$

$$\begin{aligned} d^2(fg)(x)(X,Y) &= d^2f(x)(X,Y).g(x) + df(x)(X)dg(x)(Y) \\ &+ df(x)(Y)dg(x)(X) + f(x)d^2g(x)(X,Y) \end{aligned}$$

$$\begin{aligned} d^2(fg)(x)(X,Y) &= d^2f(x)(X,Y).g(x) + \langle \nabla f(x), X\rangle\langle \nabla g(x), Y\rangle \\ &+ \langle \nabla f(x), Y\rangle\langle \nabla g(x), X\rangle + f(x)d^2g(x)(X,Y) \end{aligned}$$

Poisson bracket not given by a Poisson tensor

Queer tangent vector [Kriegl-Michor]

Identify the second derivative of f (resp. g) with an operator A (resp. B)

$$d^2(f)(x)(X,Y) = \langle X, AY\rangle \quad \text{and} \quad d^2(g)(x)(X,Y) = \langle X, BY\rangle$$

we get

$$\begin{aligned} d^2(fg)(x)(X,Y) &= \langle X, AY\rangle.g(x) + \langle X, \nabla f(x)\rangle\langle \nabla g(x), Y\rangle \\ &+\langle Y, \nabla f(x)\rangle\langle \nabla g(x), X\rangle + f(x).\langle X, AY\rangle \end{aligned}$$

$$\begin{aligned} d^2(fg)(x)(X,Y) &= \langle X, g(x).AY + \nabla f(x)\langle \nabla g(x), Y\rangle \\ &+\langle Y, \nabla f(x)\rangle \nabla g(x) + f(x).AY\rangle \end{aligned}$$

Hence $d^2(fg)(x)$ is identified with the operator

$$d^2(fg)(x) \quad = A.g(x) + \nabla f(x)\nabla g(x)^T + \nabla g(x)\nabla f(x)^T + f(x).B$$

Poisson bracket not given by a Poisson tensor

Queer tangent vector [Kriegl-Michor]

Note that $\nabla f(x)\nabla g(x)^T$ and $\nabla g(x)\nabla f(x)^T$ are rank 1 operators, hence compact.

$$\begin{aligned} D_x(fg) &= \ell(d^2(fg)(x)) \\ &= \ell\left(A.g(x) + \nabla f(x)\nabla g(x)^T + \nabla g(x)\nabla f(x)^T + f(x).B\right) \end{aligned}$$

But $\ell(\nabla f(x)\nabla g(x)^T) = 0$ and $\ell(\nabla g(x)\nabla f(x)^T) = 0$, hence

$$D_x(fg) = D_x f.g(x) + f(x).D_x g$$

Poisson bracket not given by a Poisson tensor

Queer Poisson bracket [Beltita-Golinski-Tumpach]

Consider $\mathscr{M} = \mathscr{H} \times \mathbb{R}$. Denote points of $\mathscr{M}$ as (x, λ).
Consider D_x acting on $f \in C^\infty_x(\mathscr{H})$ by $D_x(f) = \ell(d^2(f)(x))$.
Then $\{\cdot, \cdot\}$ defined by

$$\{f, g\}(x, \lambda) := D_x\left(f(\cdot, \lambda)\right) \frac{\partial g}{\partial \lambda}(x, \lambda) - \frac{\partial f}{\partial \lambda}(x, \lambda) D_x\left(g(\cdot, \lambda)\right)$$

can not be represented by a bivector field $\Pi \;: T^*\mathscr{M} \times T^*\mathscr{M} \to \mathbb{R}$.

The Hamiltonian vector field associated to $h(x, \lambda) = -\lambda$ is

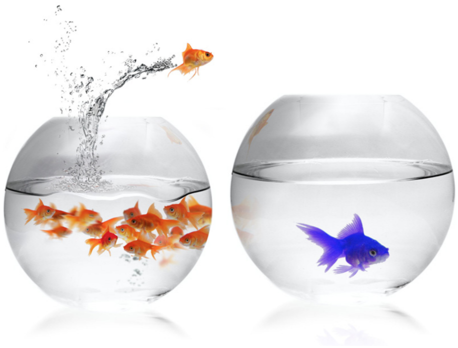
$$X_h = \{h, \cdot\} = D_x$$

Reference :

D. Beltita, T. Golinski, A.B.Tumpach, *Queer Poisson Brackets*, Journal of Geometry and Physics

Banach Poisson–Lie groups

Poisson–Lie group = Lie group with compatible Poisson structure



Poisson–Lie groups in the finite-dimensional case

connected simply connected Poisson–Lie groups

$$\Updownarrow$$

Lie-bialgebras

$$\Updownarrow$$

Manin triples

Poisson–Lie groups in the infinite-dimensional case

Banach Poisson–Lie group + restrictions on Poisson bracket

$\Downarrow$

Banach Lie-bialgebra + Banach Lie-Poisson space

$\Updownarrow$

Manin triple

Poisson–Lie groups in the infinite-dimensional case

Definition of a Manin triple

A **Banach Manin** triple consists of a triple of Banach Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ over a field $\mathbb{K}$ and a **non-degenerate symmetric bilinear** continuous map $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ such that

1. the bilinear map $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is invariant with respect to the bracket $[\cdot, \cdot]_{\mathfrak{g}}$ of $\mathfrak{g}$, i.e.
$$\langle [x, y]_{\mathfrak{g}}, z \rangle_{\mathfrak{g}} + \langle y, [x, z]_{\mathfrak{g}} \rangle_{\mathfrak{g}} = 0, \quad \forall x, y, z \in \mathfrak{g}; \tag{1}$$
2. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as Banach spaces;
3. both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Banach Lie subalgebras of $\mathfrak{g}$;
4. both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic with respect to the bilinear map $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.

Poisson–Lie groups in the infinite-dimensional case

Example of a Manin triple

$\mathfrak{u}(n)$ = Lie-algebra of the unitary group $\mathrm{U}(n)$
= space of skew-symmetric matrices

$\mathfrak{b}(n)$ = Lie-algebra of the Borel group $\mathrm{B}(n, \mathbb{C})$
= space of upper triangular matrices with real coef. on diagonal

Then the space $M(n, \mathbb{C}) = \mathfrak{gl}(n, \mathbb{C})$ of all complex matrices is a Manin triple :

$$M(n, \mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{b}(n)$$

with non-degenerate symmetric bilinear continuous map $\langle \cdot, \cdot \rangle$ given by

$$\langle A, B \rangle = \operatorname{Im} \operatorname{Tr}(AB) = \text{ imaginary part of trace}(AB)$$

Bruhat-Poisson structure of finite-dimensional Grassmannians

Proposition :

- $\mathrm{U}(n)$ and $\mathrm{B}(n, \mathbb{C})$ are dual Poisson-Lie groups
- the Grassmannians $\mathrm{Gr}(p, n) = \mathrm{U}(n)/(U(p) \times U(n-p))$ are Poisson homogeneous spaces
- the right action of $\mathrm{B}(n, \mathbb{C})$ on $\mathrm{Gr}(p, n)$ is a Poisson map
- the symplectic leaves of $\mathrm{Gr}(p, n)$ are the orbits under the action of $\mathrm{B}(n, \mathbb{C})$

Reference :

J.-H. Lu, A. Weinstein, *Poisson Lie groups, Dressing Transformations, and Bruhat Decompositions*, Journal of Differential Geometry, 1990.

Poisson–Lie groups in the infinite-dimensional case

Counterexample of a Manin triple

$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_- =$ separable complex Hilbert space, $\dim \mathscr{H}_\pm = \infty$

$$\mathrm{U}_{1,2} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(\mathscr{H}), A, D \in L^1, B, C \in L^2 \right\}$$

$\mathfrak{u}_{1,2} =$ Lie-algebra of the unitary group $\mathrm{U}_{1,2}$
$=$ space of skew-symmetric matrices with diagonal block trace class and non-diagonal block Hilbert-Schmidt

$$\mathrm{U}_{\mathrm{res}} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(\mathscr{H}), B, C \in L^2 \right\}$$

$\mathrm{B}_{\mathrm{res}} =$ invertible triangular operators $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$
with strictly positive coefficients on the diagonal and $B \in L^2$

$\mathfrak{b}_{\mathrm{res}} =$ Lie-algebra of the $\mathrm{B}_{\mathrm{res}}$
$=$ space of upper triangular operators with real coef. on diagonal and upper-right block Hilbert-Schmidt

$\mathfrak{b}_{1,2} =$ space of upper triangular operators with real coef. on diagonal upper-right block Hilbert-Schmidt and diagonal block trace-class

Poisson–Lie groups in the infinite-dimensional case

Counterexample of a Manin triple

$L_{\mathrm{res}}(\mathscr{H}) := \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), B \text{ and } C \text{ Hilbert-Schmidt}\}$
$L_{1,2}(\mathscr{H}) := \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), A \text{ and } C \text{ Trace class}, B \text{ and } C \text{ Hilbert-Schmidt}\}$

Then

- $\mathfrak{u}_{1,2} \oplus \mathfrak{b}_{1,2} \subsetneq L_{1,2}(\mathscr{H})$
- the map $\langle \cdot, \cdot \rangle$ defined on $\mathfrak{u}_{1,2} \times \mathfrak{b}_{\mathrm{res}}$ by

$$\langle A, B \rangle = \operatorname{Im} \operatorname{Tr}(AB) = \text{ imaginary part of (rest) trace}(AB)$$

is a non-degenerate symmetric bilinear continuous map, in other word a duality pairing, but $\mathfrak{u}_{1,2} \oplus \mathfrak{b}_{\mathrm{res}}$ can NOT be maid to a Manin triple.

Example of bounded operator with unbounded triangular truncation [Davidson, Nest Algebras]

$$
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \frac{1}{n-1} & \frac{1}{n} & \ddots \\
\ddots & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \frac{1}{n-1} & \ddots \\
\ddots & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \ddots \\
\ddots & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & \ddots \\
\ddots & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \ddots \\
\ddots & -\frac{1}{n-1} & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \ddots \\
\ddots & -\frac{1}{n} & -\frac{1}{n-1} & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
$$

- the triangular truncation is unbounded on the Banach space of trace class operators
- Does there exists a trace class operator whose triangular truncation is not trace class?

Theorem [T] :

- $\mathrm{U}_{\mathrm{res}}(\mathscr{H})$ and $\mathrm{B}_{\mathrm{res}}(\mathscr{H})$ are Banach Poisson–Lie groups
- The restricted Grassmannian

$$\mathrm{Gr}_{\mathrm{res}}(\mathscr{H}) = \mathrm{U}_{\mathrm{res}}(\mathscr{H})/\,\mathrm{U}(\mathscr{H}_+) \times \mathrm{U}(\mathscr{H}_-)$$

  is a Poisson homogeneous space
- the right action of $\mathrm{B}_{\mathrm{res}}(\mathscr{H})$ on $\mathrm{Gr}_{\mathrm{res}}(\mathscr{H})$ is a Poisson map
- the symplectic leaves of $\mathrm{Gr}_{\mathrm{res}}(\mathscr{H})$ are the orbits of $\mathrm{B}_{\mathrm{res}}(\mathscr{H})$.

Reference :

A.B.Tumpach, *Banach Poisson Lie groups, and Bruhat-Poisson structure of the restricted Grasssmannian*, Communications in Mathematical Physics, 2020.

Poisson manifold modelled on a non-separable Banach space

Problems :

(1) no bump functions available (norm not even $\mathscr{C}^1$ away from the origin)

(2) Leibniz rule does not imply existence of Poisson tensor (there exists derivation of order greater then 1)

(3) existence of Hamiltonian vector field is not automatic

Definition of a Banach Poisson manifold

Definition of a Poisson tensor :
M Banach manifold, $\mathbb{F}$ a subbundle of T^*M in duality with TM.
π smooth section of $\Lambda^2\mathbb{F}^*(\mathbb{F})$ is called a Poisson tensor on M with respect to $\mathbb{F}$ if :

1. for any closed local sections α, β of $\mathbb{F}$, the differential $d\left(\pi(\alpha,\beta)\right)$ is a local section of $\mathbb{F}$;
2. (Jacobi) for any closed local sections α, β, γ of $\mathbb{F}$,

$$\pi\left(\alpha, d\left(\pi(\beta,\gamma)\right)\right) + \pi\left(\beta, d\left(\pi(\gamma,\alpha)\right)\right) + \pi\left(\gamma, d\left(\pi(\alpha,\beta)\right)\right) = 0.$$

Definition of a Poisson Manifold :
A Banach Poisson manifold is a triple $(M, \mathbb{F}, \pi)$ consisting of a smooth Banach manifold M, a subbundle $\mathbb{F}$ of the cotangent bundle T^*M in duality with TM, and a Poisson tensor π on M with respect to $\mathbb{F}$.

Banach symplectic manifold

Any Banach symplectic manifold (M, ω) is naturally a generalized Banach Poisson manifold $(M, \mathbb{F}, \pi)$ with

1. $\mathbb{F} = \omega^\sharp(TM)$;
2. $\pi \ : \omega^\sharp(TM) \times \omega^\sharp(TM) \to \mathbb{R}$ defined by $(\alpha, \beta) \mapsto \omega(X_\alpha, X_\beta)$ where X_α and X_β are uniquely defined by $\alpha = \omega(X_\alpha, \cdot)$ and $\beta = \omega(X_\beta, \cdot)$.

Definition of Banach Poisson-Lie groups

Definition : A Banach Poisson-Lie group B is a Banach Lie group equipped with a Banach Poisson manifold structure such that the group multiplication $m : B \times B \to B$ is a Poisson map, where $B \times B$ is endowed with the product Poisson structure.

Proposition : Let B be a Banach Lie group and $(B, \mathbb{B}, \pi)$ a Banach Poisson structure on B. Then B is a Banach Poisson-Lie group if and only if

1. $\mathbb{B}$ is invariant under left and right multiplications by elements in B,
2. the subspace $\mathfrak{u} := \mathbb{B}_e \subset \mathfrak{b}^*$, where e is the unit element of B, is invariant under the coadjoint action of B on $\mathfrak{b}^*$ and the map

$$\begin{array}{cccc} \pi_r & : & B & \to & \Lambda^2\mathfrak{u}^*(\mathfrak{u}) \\ & & g & \mapsto & R^{**}_{g^{-1}}\pi_g, \end{array}$$

   is a 1-cocycle on B with respect to the coadjoint representation of B in $\Lambda^2\mathfrak{u}^*(\mathfrak{u})$.

Banach Lie bialgebras

Definition : Let $\mathfrak{b}$ be a Banach Lie algebra, and a duality pairing $\langle \cdot, \cdot \rangle_{\mathfrak{b},\mathfrak{u}}$ between $\mathfrak{b}$ and a normed vector space $\mathfrak{u}$. One says that $\mathfrak{b}$ is a Banach Lie bialgebra with respect to $\mathfrak{u}$ if

(1) $\mathfrak{b}$ acts continuously by coadjoint action on $\mathfrak{u}$.

(2) there is a 1-cocycle $\theta \; : \mathfrak{b} \to \Lambda^2\mathfrak{u}^*(\mathfrak{u})$ with respect to the adjoint representation of $\mathfrak{b}$ on $\Lambda^2\mathfrak{u}^*(\mathfrak{u})$, i.e. satisfying

$$\begin{aligned}\theta\left([x,y]\right)(\alpha,\beta) = \; & \theta(y)(\mathrm{ad}^*_x\alpha,\beta) + \theta(y)(\alpha,\mathrm{ad}^*_x\beta) \\ & -\theta(x)(\mathrm{ad}^*_y\alpha,\beta) - \theta(x)(\alpha,\mathrm{ad}^*_y\beta)\end{aligned}$$

where $x, y \in \mathfrak{b}$ and $\alpha, \beta \in \mathfrak{u}$.

Banach Lie bialgebras versus Manin triple

Definition : [A. A. Odzijewicz, T. Ratiu, 2003]
We will say that $\mathfrak{b}$ is a Banach Lie-Poisson space with respect to $\mathfrak{u}$ if $\mathfrak{u}$ is in duality with $\mathfrak{b}$ and is a Banach Lie algebra $(\mathfrak{u}, [\cdot,\cdot]_{\mathfrak{u}})$ which acts continuously on $\mathfrak{b}$ by coadjoint action.

Theorem [T] :
Consider two Banach Lie algebras $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}})$ and $(\mathfrak{u}, [\cdot,\cdot]_{\mathfrak{u}})$ in duality. Denote by $\mathfrak{g}$ the Banach space $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}$ with norm $\|\cdot\|_{\mathfrak{g}} = \|\cdot\|_{\mathfrak{b}} + \|\cdot\|_{\mathfrak{u}}$. The following assertions are equivalent.

(1) $\mathfrak{b}$ is a Banach Lie-Poisson space and a Banach Lie bialgebra with respect to $\mathfrak{u}$;

(2) $(\mathfrak{g}, \mathfrak{b}, \mathfrak{u})$ is a Manin triple for the natural non-degenerate symmetric bilinear map

$$\begin{array}{rccc} \langle\cdot,\cdot\rangle_{\mathfrak{g}} : & \mathfrak{g}\times\mathfrak{g} & \to & \mathbb{K} \\ & (x,\alpha)\times(y,\beta) & \mapsto & \langle x,\beta\rangle_{\mathfrak{b},\mathfrak{u}} + \langle y,\alpha\rangle_{\mathfrak{b},\mathfrak{u}}. \end{array}$$

Banach Lie–Poisson spaces

Theorem [T]

The Banach Lie algebra $\mathfrak{u}_{1,2}(\mathscr{H})$ is not a Banach Lie–Poisson space with respect to $\mathfrak{b}_{\mathrm{res}}(\mathscr{H})$.

Consequently there is no Banach Manin triple structure on the triple of Banach Lie algebras $(\mathfrak{b}_{\mathrm{res}}(\mathscr{H}) \oplus \mathfrak{u}_{1,2}(\mathscr{H}), \mathfrak{b}_{\mathrm{res}}(\mathscr{H}), \mathfrak{u}_{1,2}(\mathscr{H}))$ for the duality pairing given by the imaginary part of the trace.

Theorem [T] :

Let $(G_+, \mathbb{F}, \pi)$ be a Banach Poisson–Lie group. Then $\mathfrak{g}_+$ is a Banach Lie bialgebra with respect to $\mathfrak{g}_-$. The Lie bracket in $\mathfrak{g}_-$ is given by

$$[\alpha_1, \beta_1]_{\mathfrak{g}_-} := T_e\Pi_r(\cdot)(\alpha_1, \beta_1) \in \mathfrak{g}_- \subset \mathfrak{g}_+^*, \quad \alpha_1, \beta_1 \in \mathfrak{g}_- \subset \mathfrak{g}_+^*, \tag{2}$$

where $\Pi_r := R_{g^{-1}}^{**}\pi \ : G_+ \to \Lambda^2\mathfrak{g}_-^*$, and $T_e\Pi_r : \mathfrak{g}_+ \to \Lambda^2\mathfrak{g}_-^*$ denotes the differential of Π_r at the unit element $e \in G_+$.

Theorem [T] :

Let $(G_+, \mathbb{F}, \pi)$ be a Banach Poisson–Lie group.If the map $\pi^\sharp \ : \mathbb{F} \to \mathbb{F}^*$ defined by $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$ takes values in $TG_+ \subset \mathbb{F}^*$, then $\mathfrak{g}_+$ is a Banach Lie–Poisson space with respect to $\mathfrak{g}_- := \mathbb{F}_e$.

Poisson–Lie groups in the infinite-dimensional case

Banach Poisson–Lie group G + $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$ takes values in TG

$\Downarrow$

Banach Lie-bialgebra + Banach Lie-Poisson space

$\Updownarrow$

Manin triple

Poisson–Lie groups in the infinite-dimensional case

THANK YOU FOR YOUR ATTENTION !

COME AND VISIT VIENNA !

FWF Grant I 5015-N : **Banach Poisson–Lie Groups, Integrable systems, and extension to the Fréchet context**

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