Infinite-dimensional Geometry : Theory and Applications

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Outline

Lecture 1

Basics notions in infinite-dimensional geometry

Lecture 2

Inverse Function Theorems : Banach version and Nash-Moser version

Lecture 3

Some pathologies of infinite-dimensional geometry

Outline

Lecture 1 : Basics notions in infinite-dimensional geometry

1. Manifolds : model spaces and their smooth functions
2. Tangent bundles, Cotangent bundles and their relatives
3. Examples from Geometry, Shape Analysis and Gauge Theory
4. Key Tools from Functional Analysis

Lecture 2

Inverse Function Theorems : Banach version and Nash-Moser version

Lecture 3

Some pathologies of infinite-dimensional geometry

Outline

Lecture 1

Basics notions in infinite-dimensional geometry

Lecture 2

Inverse Function Theorems : Banach version and Nash-Moser version

1. The Banach version and its proof
2. Tame category and Nash-Moser version
3. Toolkit to use the Nash-Moser version
4. Ideas of Nash-Moser's proof
5. Some applications

Lecture 3

Some pathologies of infinite-dimensional geometry

Why infinite-dimensional geometry?

At the backstage of finite-dimensional geometry

- existence of geodesics on a finite-dimensional manifold is an infinite-dimensional phenomenon
  - initial value problem or shooting : geodesic is a solution of a Cauchy problem, i.e. a fixed point of a contraction in an appropriate infinite-dimensional space of curves
  - 2 boundary value problem: geodesic is a curve minimising an energy functional on a infinite-dimesnional space of curves
- natural objects on a finite-dimensional manifold are elements of an infinite-dimensional space (vector fields, Riemannian metrics, mesures...)
- Each time one want to vary the geometry of a finite-dimensional manifold, one ends up with a infinite-dimensional manifold (of Riemannian metric, of connexions, of symplectic forms....)

Why infinite-dimensional geometry?

What is not covered by these Lectures?

- **The convenient setting of global analysis**, Andreas Kriegl and Peter W. Michor, volume 53 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997.
- **Diffeological spaces**, J.-M. Souriau. Groupes différentiels. In Differential geometrical methods in mathematical physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979), volume 836 of Lecture Notes in Math., pages 91–128. Springer, Berlin, 1980.
- **Bastiani calculus on locally convex spaces**, A. Bastiani, Applications différentiables et variétés différentiables de dimension infinie, J. Analyse Math. 13 (1964):1–114.
- **Frölicher spaces**, Alfred Frölicher. Smooth structures. In Category theory (Gummersbach, 1981), volume 962 of Lecture Notes in Math., pages 69–81. Springer, Berlin, 1982.
- **Ringed spaces**, Egeileh, Michel, and Tilmann Wurzbacher. "Infinite-dimensional manifolds as ringed spaces." Publications of the Research Institute for Mathematical Sciences, vol. 53, no. 1, 2017.
- **Comparative smootheologies**, Andrew Stacey, Theory and Applications of Categories, Vol. 25, No. 4, 2011, pp. 64–117.
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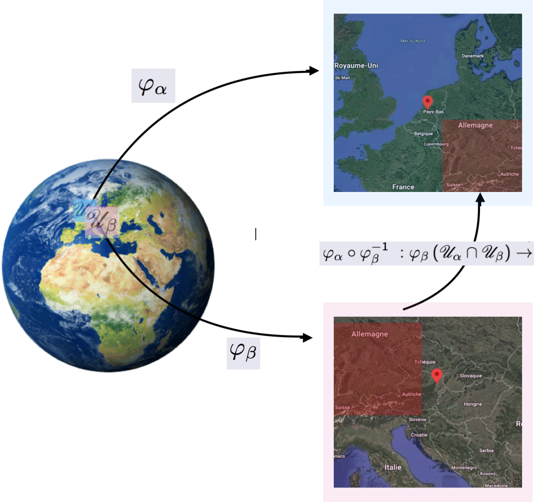
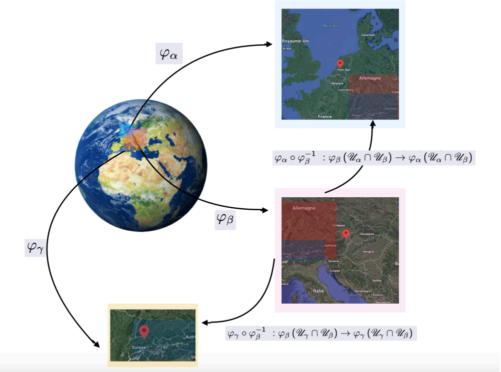
Definition of an infinite-dimensional manifold



$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

Definition of an infinite-dimensional manifold



The notion of manifold is build out from the notion of SMOOTH maps (or $\mathscr{C}^k$, or $\mathscr{C}^w$) between MODEL SPACES, the crucial condition on the set of smooth maps is the CHAIN RULE.

Charts and complete Atlas

Definition of a chart

Definition of an atlas

$\mathscr{C}^k$ equivalent atlases

Manifold = Hausdorff topological space with an equivalence class of $\mathscr{C}^k$ atlases

Charts and complete Atlas

Definition of a chart

A **chart** on a topological space $\mathscr{M}$ is a triple $(\mathscr{U}, \varphi, \mathscr{F}_\alpha)$ where $\mathscr{U}$ is an open set in $\mathscr{M}$ and ϕ an homeomorphism from $\mathscr{U}$ to an open set in a model topological vector space $\mathscr{F}_\alpha$.

Definition of an atlas

An **atlas** on a topological space $\mathscr{M}$ is a collection of charts $(\mathscr{U}_\alpha, \varphi_\alpha, \mathscr{F}_\alpha)_{\alpha\in\mathscr{I}}$ such that $\cup_{\alpha\in\mathscr{I}}\mathscr{U}_\alpha = \mathscr{M}$

Atlas of class $\mathscr{C}^k$

An **atlas** $\mathscr{A} = (\mathscr{U}_\alpha, \varphi_\alpha, \mathscr{F}_\alpha)_{\alpha\in\mathscr{I}}$ on $\mathscr{M}$ is of class $\mathscr{C}^k$ if all transition maps are $\mathscr{C}^k$-maps between the model topological vector spaces :
$\forall(\mathscr{U}_\alpha, \varphi_\alpha, \mathscr{F}_\alpha) \in \mathscr{A}$ and $(\mathscr{U}_\beta, \varphi_\beta, \mathscr{F}_\beta) \in \mathscr{A}$ such that $\mathscr{U}_\alpha \cap \mathscr{U}_\beta \neq \emptyset$
$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta\left(\mathscr{U}_\alpha \cap \mathscr{U}_\beta\right) \rightarrow \varphi_\alpha\left(\mathscr{U}_\alpha \cap \mathscr{U}_\beta\right)$ is of class $\mathscr{C}^k$

Charts and complete Atlas

$\mathscr{C}^k$ equivalent atlases

Two atlases on a topological space $\mathscr{M}$ are said to be $\mathscr{C}^k$ **equivalent** if their union is of class $\mathscr{C}^k$

Manifolds of class $\mathscr{C}^k$

A **manifold of class** $\mathscr{C}^k$ $(k \geq 0)$ is an Hausdorff topological space endowed with an equivalence class of $\mathscr{C}^k$-atlases.

Hausdorff space

A topological space $\mathscr{M}$ is said to be **Hausdorff** if for any pair of distinct points $f_0 \neq f_1$ in $\mathscr{M}$ one can find two disjoints open sets $\mathscr{U}_0 \cap \mathscr{U}_1 = \emptyset$ in $\mathscr{M}$ such that $f_0 \in \mathscr{U}_0$ and $f_1 \in \mathscr{U}_1$

Remarks

On the global level

WE WILL NOT ASSUME that a manifold $\mathscr{M}$ is

- **paracompact** (every open cover has an open refinement that is locally finite)
- **admits smooth partitions of unity**
- **second countability** (the topology has a countable base)
- **separability** (there exists a countable dense subset)
- **Lindelöf** (every open cover has a countable subcover)

On the local level

WE WILL NOT ASSUME existence of **smooth bump functions**
BUT the manifolds will be **first-countable** (every point has a countable neighbourhood basis) because our model spaces will be metrizable

Complete metric spaces

Metric space

A metric space is a space $\mathscr{M}$ endowed with a **distance function** $d : \mathscr{M} \times \mathscr{M} \to \mathbb{R}$

- $d(f_0, f_1) = 0 \Leftrightarrow f_0 = f_1$
- $d(f_0, f_1) = d(f_1, f_0)$
- $d(f_0, f_1) \leq d(f_0, f_2) + d(f_2, f_0)$

Cauchy sequence

A sequence $\{f_k\}_{k\in\mathbb{N}}$ in a metric space $\mathscr{M}$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists $N > 0$ suth that $d(f_n, f_m) < \varepsilon$ for all $n, m > N$

Complete metric space

A metric space $\mathscr{M}$ is said to be **complete** if any Cauchy sequence of elements in $\mathscr{M}$ converges

What are the Model spaces of infinite-dim. geometry?

Hilbert $\subset$ Banach $\subset$ Fréchet $\subset$ Locally Convex Spaces

Hilbert space H = **complete** vector space for the distance given by an inner product = $\langle \cdot, \cdot \rangle \; : H \times H \to \mathbb{R}^+$

- symmetric :$\langle x, y \rangle = \langle y, x \rangle$
- bilinear : $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$
- non-negative : $\langle x, x \rangle \geq 0$
- definite : $\langle x, x \rangle = 0 \Rightarrow x = 0$

$H^* = H$ (Riesz Theorem).

What are the Model spaces of infinite-dim. geometry?

Hilbert $\subset$ **Banach** $\subset$ Fréchet $\subset$ Locally Convex spaces

Banach space B = complete vector space for the distance given by a norm $= \|\cdot\| \ : B \to \mathbb{R}^+$

- triangle inequality : $\|x+y\| \leq \|x\| + \|y\|$
- absolute homogeneity : $\|\lambda x\| = |\lambda|\|x\|$.
- non-negative : $\|x\| \geq 0$
- definite : $\|x\| = 0 \Rightarrow x = 0$.

B^* = Banach space.

What are the Model spaces of infinite-dim. geometry?

Hilbert $\subset$ Banach $\subset$ **Fréchet** $\subset$ Locally Convex spaces

Fréchet space F = complete Hausdorff vector space for the distance $d\ : F \times F \to \mathbb{R}^+$ given by a countable family of semi-norms $\|\cdot\|_n$:

$$d(x,y) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n}$$

$F^* \neq$ Fréchet space if F not Banach, but locally convex
$F^{**} =$ Fréchet space.

What are the Model spaces of infinite-dim. geometry?

Hilbert $\subset$ Banach $\subset$ Fréchet $\subset$ **Locally Convex spaces**

Locally Convex spaces = Hausdorff topological vector space whose topology is given by a (possibly not countable) family of semi-norms.

References :

- **The convenient setting of global analysis**, Kriegl, Michor
- **Diffeological spaces**, Souriau
- **Bastiani calculus on locally convex spaces**, Bastiani
- **Frölicher spaces**, Frölicher
- **Ringed spaces**, Egeileh, Michel, and Wurzbacher
- **Comparative smootheologies**, Stacey

What are the smooth maps between the model spaces?

Differentiable function on $\mathbb{R}^n$

For a function $f\ : \mathscr{U} \subset \mathbb{R} \to \mathbb{R}$, there are 3 equivalent notions of been **differentiable** at $x \in \mathscr{U}$

- $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists and is finite
- $\exists L$ such that $\lim_{h\to 0} \frac{f(x+h)-f(x)-Lh}{h} = 0$
- there exists a function $g\ : \mathscr{U} \subset \mathbb{R} \to \mathbb{R}$, such that $f(x+h) = f(x) + f'(x)h + g(h)$ and $\lim_{h\to 0} \frac{g(h)}{h} = 0$

Remark

On $\mathbb{R}$, a differentiable function is automatically continuous

In the Banach context

Fréchet differentiability in the Banach context

Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be two Banach spaces. A map $P : \mathscr{U} \subset \mathscr{B}_1 \to \mathscr{B}_2$ is **Fréchet differentiable** at $f_0 \in \mathscr{B}_1$ if there exists a **continuous linear operator** $DP(f_0) : \mathscr{B}_1 \to \mathscr{B}_2$ such that

$$P(f_0 + h) = P(f_0) + DP(f_0)(h) + \|h\|_1 \cdot \varepsilon(h) \quad \text{with} \quad \lim_{h\to 0} \|\varepsilon(h)\|_2 = 0$$

Remark

No continuity is assumed in the definition of Fréchet differentiability, but **Fréchet differentiable at** $f_0 \Rightarrow$ **continuous at** f_0

In the Banach context

$\mathscr{C}^1$ in the Banach context

A map $P : \mathscr{U} \subset \mathscr{B}_1 \to \mathscr{B}_2$ between Banach spaces is $\mathscr{C}^1$ if it is Fréchet differentiable on $\mathscr{U}$ and the derivative DP is continuous as a map from $\mathscr{U}$ into the Banach space $L_c(\mathscr{B}_1, \mathscr{B}_2)$ of continuous linear operators from $\mathscr{B}_1$ to $\mathscr{B}_2$

Smooth maps between Banach spaces

By induction one defines the notion of smooth maps on Banach spaces.

In the Fréchet context

Directional derivative

Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two Fréchet spaces and $P : \mathscr{U} \subset \mathscr{F}_1 \to \mathscr{F}_2$ a **continuous** non-linear map. P admits a **derivative at** f_0 **in the direction** of $h \in \mathscr{F}_1$ if the following limit exists

$$DP(f_0)(h) = \lim_{t \to 0} \frac{P(f_0 + th) - P(f_0)}{t}$$

One says that P is differentiable at f_0 if it admits directional derivatives in every direction $h \in \mathscr{F}_1$

$\mathscr{C}^1$ in the Fréchet context

A map $P : \mathscr{U} \subset \mathscr{F}_1 \to \mathscr{F}_2$ between Fréchet spaces is $\mathscr{C}^1$ if it is differentiable in $\mathscr{U}$ and the derivative DP is continuous as a map from $\mathscr{U} \times \mathscr{F}_1$ into $\mathscr{F}_2$

Comparaison of the two notions of $\mathscr{C}^1$-maps

Remarks

no linearity assumed but if P is $\mathscr{C}^1$ then $DP(f)h$ is always linear in h

On a Banach space

- $\mathscr{C}^1$ in the Banach context $\Leftrightarrow$ $\mathscr{C}^1$ in the Fréchet context
- $\mathscr{C}^2$ in the Fréchet context $\Leftrightarrow$ $\mathscr{C}^1$ in the Banach context [Keller]

Taylor formula

If $P\ :\mathscr{U}\subseteq\mathscr{F}\to G$ is $\mathscr{C}^2$ and if the path connecting f and $f+h$ lies in $\mathscr{U}$ then

$$P(f+h)=P(f)+DP(f)(h)+\int_0^1(1-t)D^2P(f+th)(h,h)dt$$

What is the Tangent vector ?

Kinetic tangent vector

A Kinetic tangent vector $X \in T_{f_0}\mathscr{M}$ is an equivalence classes of curves $c(t)$, $c(0) = f_0$, where $c_1 \sim c_2$ if they have the same derivative at 0 in a chart.

Remark : Kinetic tangent vectors can be identified in a chart with elements of the model space.

Operational tangent vector

An operational tangent vector at $f_0 \in \mathscr{M}$ is a linear map $D : C^\infty_{f_0}(\mathscr{M}) \to \mathbb{R}$ satisfying Leibniz rule :

$$D(PQ)(f_0) = DP\ Q(f_0) + P(f_0)\ DQ$$

Remark : Even on a Hilbert space there exists operational tangent vectors which are not Kinetic tangent vectors (cf Lecture 3)

What is a cotangent vector ?

(Kinetic) cotangent vector

A (kinetic) cotangent vector $F \in T^*_{f_0}\mathscr{M}$ is an continuous linear functional on the space of kinetic tangent vectors

Remark : Kinetic cotangent vectors can be identified in a chart with elements of the **continuous dual of the model space**. If the model space is a Fréchet space which is not a Banach space, the continuous dual of the model space is not a Fréchet space

What are the tangent bundle and cotangent bundles?

Tangent bundle

On a Fréchet manifold $\mathscr{M}$ the set $T\mathscr{M}$ of all kinetic tangent vectors has a natural structure of smooth Fréchet manifold with canonical projection $\pi\ : T\mathscr{M} \to \mathscr{M}$, $X \in T_{f_0}\mathscr{M} \mapsto f_0$

Cotangent bundle

On a Banach manifold $\mathscr{M}$ the set $T^*\mathscr{M}$ of all (kinetic) cotangent vectors has a natural structure of smooth Banach manifold with canonical projection $\pi\ : T^*\mathscr{M} \to \mathscr{M}$, $F \in T^*_{f_0}\mathscr{M} \mapsto f_0$

What is the problem with tensor products?

Universal property for algebraic tensor product

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two $\mathbb{K}$-vector spaces. The **algebraic tensor product** $\mathfrak{g}_1 \otimes_a \mathfrak{g}_2$ of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is the unique (up to isomorphim of $\mathbb{K}$-vector spaces) $\mathbb{K}$-vector space such that there exists a bilinear mapping

$$B : \mathfrak{g}_1 \times \mathfrak{g}_2 \to \mathfrak{g}_1 \otimes_a \mathfrak{g}_2$$

having the following **universal property :**

If $B_1 : \mathfrak{g}_1 \times \mathfrak{g}_2 \to \mathfrak{g}$ is any bilinear mapping into a $\mathbb{K}$-vector space $\mathfrak{g}$, then there exists a unique linear mapping $L : \mathfrak{g}_1 \otimes_a \mathfrak{g}_2 \to \mathfrak{g}$ such that $B_1 = L \circ B$.

Remark : The universal property implies in particular that the algebraic dual of the algebraic tensor product is the $\mathbb{K}$-vector space of $\mathbb{K}$-valued bilinear maps on $\mathfrak{g}_1 \times \mathfrak{g}_2$.

What is the problem with tensor products?

Grothendiek lists 14 different norms on tensor products of Banach spaces

Which norm on $\mathfrak{g}_1 \otimes_a \mathfrak{g}_2$ to complete it into a Banach space?
the projective cross norm?
the injective cross norm?
or one of the 12 others?

Example :

For an Hilbert space $\mathfrak{h}$ and its continous dual $\mathfrak{h}^*$, the injective tensor product of $\mathfrak{h}^*$ and $\mathfrak{h}$ is the Banach space of compact operators on $\mathfrak{h}$, whereas the projective tensor product is the Banach space of trace class operators on $\mathfrak{h}$.

Continuous Multilinear maps

In the Banach case, continuous multilinear maps forms a Banach space

For Banach spaces $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ and $\mathfrak{h}$, the space

$$\mathrm{L}^k(\mathfrak{g}_1, \mathfrak{g}_2, \ldots \mathfrak{g}_k; \mathfrak{h})$$

of continuous k-multilinear maps from the product Banach space $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ to the Banach space $\mathfrak{h}$ is itself a Banach space

Symmetric Multilinear maps

For any Banach space $\mathfrak{g}$, a multilinear map $\mathbf{t} \in L^k(\mathfrak{g}, \ldots, \mathfrak{g}; \mathbb{K})$ is said to be **symmetric** if and only if

$$\mathbf{t}(e_1, \ldots, e_k) = \mathbf{t}(e_{\sigma(1)}, \ldots, e_{\sigma(k)})$$

for any $e_1, \ldots e_k$ in $\mathfrak{g}$ and any permutation σ in the group $\mathscr{S}(k)$ of all permutations of $\{1, \ldots, k\}$

Continuous Multilinear maps

Skew-symmetric Multilinear maps

For any Banach space $\mathfrak{g}$, a multilinear map $\mathbf{t} \in L^k(\mathfrak{g}, \ldots, \mathfrak{g}; \mathbb{K})$ is said to be **skew-symmetric** if and only if

$$\mathbf{t}(e_1, \ldots, e_k) = \mathrm{sign}(\sigma)\mathbf{t}(\mathrm{e}_{\sigma(1)}, \ldots, \mathrm{e}_{\sigma(\mathrm{k})})$$

for any $e_1, \ldots e_k$ in $\mathfrak{g}$ and any permutation σ of $\{1, \ldots, k\}$, where $\mathrm{sign}(\sigma)$ denotes the signature of σ.

The space $S^k\mathfrak{g}^*$ consisting of **symmetric multilinear maps** on a Banach space $\mathfrak{g}$ is a closed subspace of $L^k(\mathfrak{g}, \ldots, \mathfrak{g}; \mathbb{K})$, hence a Banach space

The space $\Lambda^k\mathfrak{g}^*$ consisting of **skew-symmetric multilinear maps** on $\mathfrak{g}$ is a closed subspace of $L^k(\mathfrak{g}, \ldots, \mathfrak{g}; \mathbb{K})$, hence a Banach space

Continuous Multilinear maps

Symmetric Multilinear maps on a Banach manifold

If $\mathscr{M}$ is a Banach manifold, the space $S^k T^*\mathscr{M}$ of **symmetric multilinear maps** on $T\mathscr{M}$ has a natural structure of Banach manifold, and of vector bundle over $\mathscr{M}$

A section $g \; : \mathscr{M} \to T^*\mathscr{M}$ is called a **symmetric tensor** on $\mathscr{M}$.

Skew-symmetric Multilinear maps on a Banach manifold

If $\mathscr{M}$ is a Banach manifold, the space $\Lambda^k T^*\mathscr{M}$ of **skew-symmetric multilinear maps** on $T\mathscr{M}$ has a natural structure of Banach manifold, and of vector bundle over $\mathscr{M}$

A section $\omega \; : \mathscr{M} \to T^*\mathscr{M}$ is called a **skew-symmetric tensor** on $\mathscr{M}$.

What is an infinite-dimensional Lie group ?

Fréchet Lie groups

A Fréchet Lie group is a Fréchet manifold $\mathscr{G}$ with a group structure such that the multiplication map m and the inverse map inv are smooth

$$m \;: \mathscr{G} \times \mathscr{G} \to \mathscr{G}, m(g, h) = gh$$

$$\mathrm{inv} \;: \mathscr{G} \to \mathscr{G}, \mathrm{inv}(g) = g^{-1}$$

Examples

Fréchet Lie groups

- The **group of diffeomorphisms** $\mathscr{D}(M)$ of a compact manifold M
- The **group of volume preserving diffeomorphisms** $\mathscr{D}_{\mathrm{vol}}(M)$ of a compact Riemannian manifold M

Reference

- R.S. Hamilton, *The inverse function Theorem of Nash and Moser*
- J. Milnor, *Remarks on infinite-dimensional Lie groups*
- H. Glöckner, K.H. Neeb, *Banach-Lie Quotients, Enlargibility, and Universal Complexifications*
- M. Molitor, *Remarks on the space of volume preserving embeddings*
- B. Khesin, J. Lenells, G. Misiolek, S. Preston, *Curvature of Sobolev metrics on Diffeomorphism groups*

Examples from Geometry

Spheres

The sphere in a Hilbert space is a smooth Hilbert manifold

Remark : The sphere in a Banach space is not smooth unless the Banach space is an Hilbert space

Linear Grassmannians

The projective space of an Hilbert space is a Hilbert manifold
The Grassmannian of p-dimensional subspaces in an Hilbert space is Hilbert manifold (p finite)
The Grassmannian of subspaces in an Hilbert space with infinite dimension and codimension is a Banach manifold

Examples from Geometry

Restricted Linear Grassmannians

Let $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ be a decomposition of an Hilbert space into the sum of two closed infinite-dimensional orthogonal subspaces. The restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}(\mathscr{H})$ denotes the set of all closed subspaces W of $\mathscr{H}$ such that the orthogonal projection $p_- : W \to \mathscr{H}_-$ belongs to some given Schatten class (compact operator, Hilbert-Schmidt, trace class operators,...), then one obtain a Grassmannian manifold modelled on the space of operator from $\mathscr{H}_-$ to $\mathscr{H}_+$ in this given Schatten class.

Reference

- Pressley, Segal, *Loop spaces*
- T. Golinski, A. Odzijewicz, *Hierarchy of Hamilton equations on Banach Lie-Poisson spaces related to restricted Grassmannian*
- E. Andruchow, G. Larotonda, *Hopf-Rinow Theorem in the Sato Grassmannian*
- A.B.Tumpach, *Banach Poisson-Lie groups and the Bruhat-Poisson structure of the restricted Grassmannian*

Examples from Geometry

Manifolds of maps

The space of smooth maps from a compact manifold into a finite-dimensional manifold is a Fréchet manifold

Space of sections

The space of smooth section of a finite-dimensional vector bundle over a compact manifold is a Fréchet manifold

Examples from Geometry

Non-linear Grassmannians and non-linear Flags

The space of embeddings $N \hookrightarrow M$ from a compact manifold N into a finite-dimensional manifold M is a Fréchet manifold. More generally the space of flags $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_k \subset M$ is a Fréchet manifold

Reference

- M. Bauer, M. Bruveris, P.W. Michor, *Overview of the Geometries of Shape Spaces and Diffeomorphism Groups*
- F. Gay-Balmaz, C. Vizman, *Vortex sheets in ideal 3D fluids, coadjoint orbits, and characters*
- S. Haller, C. Vizman, *nonlinear Flag manifolds as coadjoint orbits*

Examples from Geometry

Manifolds of curves

The space of H^1 curves from $[0, 1]$ into a finite-dimensional Riemannian manifold M is a Hilbert manifold, and the critical points of the energy functional are the geodesics of M

Square Root Velocity Framework

Length one curves can be seen as points on the sphere of an Hilbert space.

Arc-length parameterized curves

The space of arc-length parameterized curves $c : [0, 1] \to \mathbb{R}^n$ is a Fréchet submanifold of the space of all parameterized curves.

Examples from Geometry

Reference

- W.P.A. Klingenberg, *Riemannian Geometry*
- S. Lahiri, D. Robinson, E. Klassen, *Precise Matching of PL Curves in RN in the Square Root Velocity Framework*
- A. Schmeding, *Manifolds of absolutely continuous curves and the square root velocity framework*
- E. Celledoni, S. Eidnes, A. Schmeding, *Shape analysis on homogeneous spaces: a generalised SRVT framework*
- S. Preston, A.B.Tumpach, *Quotient Elastic Metrics on the manifold of arc-length parameterized plane curves*

Shape spaces are non-linear manifolds

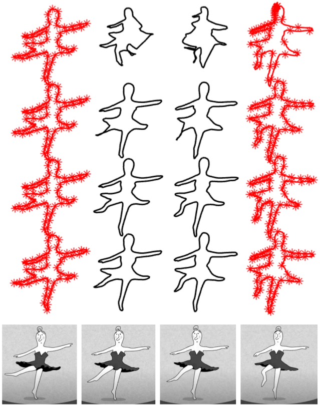
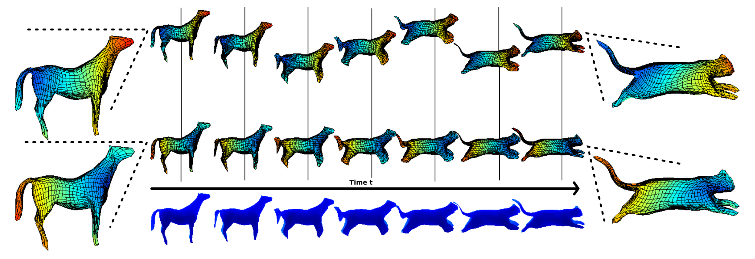


Figure: First line : linear interpolation between some parameterized ballerinas, second line : linear interpolation between arc-length parameterized ballerinas, third line : geodesic on a hilbert sphere, fourth line : improvement of third line, fifth line : ground truth

Examples from Shape Analysis



Pre-shape space $\mathscr{F} := \{f \text{ embedding } : \mathbb{S}^2 \to \mathbb{R}^3\} \subset \mathscr{C}^\infty(\mathbb{S}^2, \mathbb{R}^3)$
Shape space $\mathscr{S} :=$ 2-dimensional submanifolds of $\mathbb{R}^3$

What about genus 0 surfaces?

Question

Is there a canonical section of the fiber bundle of parameterized surfaces of genus 0?

Answer

Modulo $PSL(2,\mathbb{C})$ yes.

Question

Is this section smooth?

Answer

I don't know...

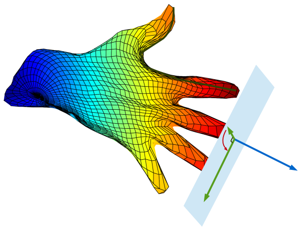


Figure: Scalar product on the tangent plan to the tip of the middle finger of a hand.

Genus-0 surfaces of $\mathbb{R}^3$ are *Riemann surfaces*. Since they are compact and simply connected, the Uniformization Theorem says that they are conformally equivalent to the unit sphere. This means that, given a spherical surface, there exists a homeomorphism, called the *uniformization map*, which preserves the angles and transforms the unit sphere into the surface. The uniformization maps are parameterized by $PSL(2, \mathbb{C})$.

Examples from Gauge Theory

The group of gauge transformations is a Fréchet Lie group acting on the space of connections on a principal bundle over a closed surface

Given a closed surface endowed with a volume form, the space of compatible Riemannian structures is an infinite-dimensional symplectic manifold. The group of volume-preserving diffeomorphisms acts by push-forward and has a group-valued momentum map. Moreover the Teichmuller space and the moduli space of Riemann surfaces can be realized as symplectic orbit reduced spaces

Reference

- S.K. Donaldson, *Nahm's Equations and the Classification of Monopoles*
- T. Diez, T. S. Ratiu, *Realizing the Teichmuller space as a symplectic quotient*
- T. Diez, T. S. Ratiu, *Group-valued momentum maps for actions of automorphism groups*

What are the Tools from Functional Analysis?

Banach-Picard fixed point Theorem or Contraction Theorem

(E, d) complete metric space
$f : E \to E$ contraction of E : $d(f(x), f(y)) \leq kd(x, y)$ where $k \in (0, 1)$

$$\Rightarrow \begin{cases} \exists\,!\ x \in E, f(x) = x \\ \forall x_0 \in E,\ \textit{the sequence } x_{n+1} = f(x_n) \textit{ converges to } x \end{cases}$$

What are the Key Tools from Functional Analysis?

Hahn-Banach Theorem

E locally convex space
$A \subset E$ a convex
$x \in E$, $x \notin \overline{A}$

$$\Rightarrow \exists \textit{ continuous functional } \ell \ : E \to \mathbb{R} \textit{ with } \ell(x) \notin \overline{\ell(A)}$$

What are the Tools from Functional Analysis?

Open mapping Theorem

$$\left\{\begin{array}{l} F \;\; \textit{Fréchet} \\ G \;\; \textit{Fréchet} \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} F \;\; \textit{webbed locally convex} \\ G \;\; \textit{inductive limit of Baire locally convex spaces} \end{array}\right.$$

$L : F \to G$ *continuous, linear, and surjective*

$\Rightarrow L$ *is open*

What are the Tools from Functional Analysis?

Cauchy-Lipschitz Theorem in the Banach case

- $\mathscr{I} \subset \mathbb{R}$ be an interval containing 0
- $\mathscr{U}$ open set of a Banach space $\mathscr{B}$
- $P \; : \mathscr{I} \times \mathscr{U} \to \mathscr{B}$

such that

- $\|P(t,f)\| \leq C \quad \forall (t,f) \in \mathscr{I} \times \mathscr{U}$
- $\|P(t,f_1) - P(t,f_0)\| \leq C' \|f_1 - f_0\| \quad \forall t \in \mathscr{I}, \forall f_0, f_1 \in \mathscr{U}$

For any $f_0 \in \mathscr{U}$ we can find a neighboorhood $\tilde{\mathscr{U}}$ of f_0 and an $\varepsilon > 0$ such that for any $f \in \tilde{\mathscr{U}}$ the Cauchy problem

$$\frac{d}{dt}\phi(t,f) = P(t,\phi(t,f))$$

has a unique solution with initial condition $\phi(0,f) = f$ on $[-\varepsilon,\varepsilon]$. Moreover if P is $\mathscr{C}^p$, $t \to \phi(t,x)$ is $\mathscr{C}^p$ for any $f \in \tilde{\mathscr{U}}$

What are the Tools from Functional Analysis?

Cauchy Theorem in the Fréchet case

Let $P : \mathscr{U} \subset \mathscr{F} \to \mathscr{F}$ be a **smooth Banach map**. Then $\forall f_0 \in \mathscr{U}$, $\exists \tilde{\mathscr{U}} \ni f_0$ and $\varepsilon > 0$ s.t. $\forall f \in \tilde{\mathscr{U}}$

$$\frac{d}{dt} f = P(f)$$

has a unique solution with initial condition $f(0) = f$ on $0 \leq t \leq \varepsilon$ depending smoothly on t and f.

What are the Tools from Functional Analysis?

Inverse function Theorem

Theorem

Let $f : \mathscr{U} \subset B_1 \rightarrow B_2$ be a $\mathscr{C}^1$-map between ***Banach*** *spaces. If $Df(a)$ is invertible at $a \in \mathscr{U}$, then there exists an open neighborhood $\mathscr{V}_a$ of $a \in \mathscr{U}$ and an open neighborhood $\mathscr{V}_{f(a)} \subset B_2$ suth that $f : \mathscr{V}_a \rightarrow \mathscr{V}_{f(a)}$ is a $\mathscr{C}^1$-diffeomorphism.*

Counterexample : $\exp : \mathrm{Lie}(\mathrm{Diff}(\mathbb{S}^1)) \rightarrow \mathrm{Diff}(\mathbb{S}^1)$ not locally onto.

Theorem (Nash-Moser)

Let $f : \mathscr{U} \subset F_1 \rightarrow F_2$ be a smooth tame map between ***Fréchet*** *spaces. Suppose that the equation for the derivative $Df(x)(h) = k$ has a unique solution $h = L(x)k$ for all $x \in \mathscr{U}$ and $\forall k \in F_2$ and that the family of inverses $L : \mathscr{U} \times F_2 \rightarrow F_1$ is a smooth tame map. Then f is locally invertible and each local inverse is a smooth tame map.*

What are the Tools from Functional Analysis?

Theorems :	Hilbert	Banach	Fréchet	Locally Convex
Banach-Picard	$\surd$	$\surd$	$\surd$	X
Open Mapping	$\surd$	$\surd$	$\surd$	F webbed G limit of Baire
Hahn-Banach	$\surd$	$\surd$	$\surd$	$\surd$
Cauchy Theorem	$\surd$	$\surd$	Hamilton	X
Inverse function	$\surd$	$\surd$	Nash-Moser	X