Stochastic processes on surfaces in 3D contact sub-Riemannian manifolds

Talk by Karen Habermann *on joint work with*
Davide Barilari, Ugo Boscain and Daniele Cannarsa

arXiv:2004.13700
(to appear in *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*)

15th International Young Researchers Workshop on Geometry, Mechanics, and Control

Setting

Setting

- M three-dimensional smooth manifold,

Setting

- M three-dimensional smooth manifold,
- D contact structure on M,

Setting

- M three-dimensional smooth manifold,
- D contact structure on M, that is,
  $D = \ker \omega$ for one-form ω on M with $\omega \wedge \mathrm{d}\omega \neq 0$,

Setting

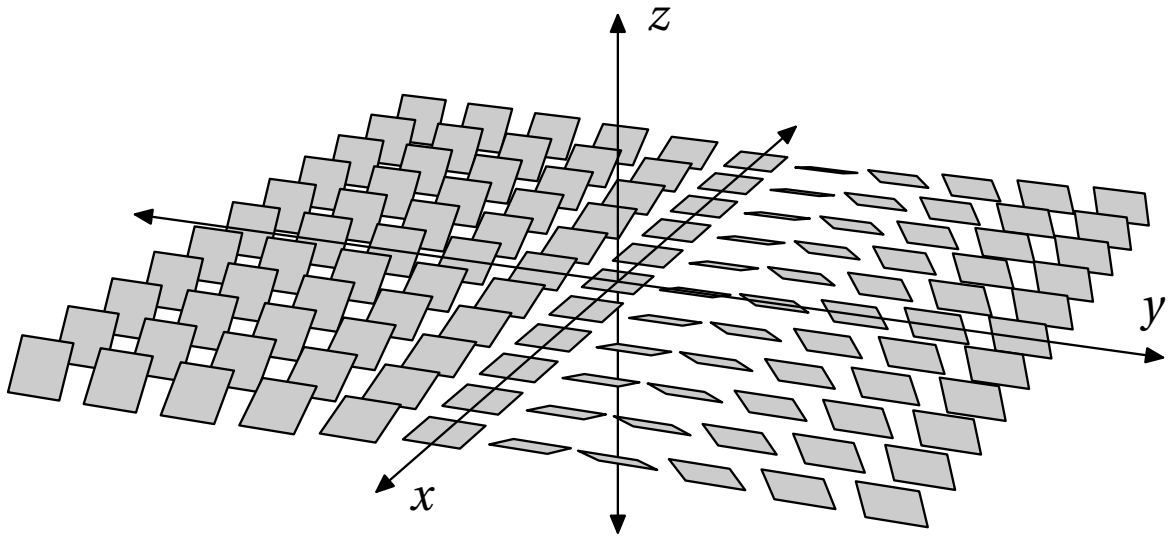
- M three-dimensional smooth manifold,
- D contact structure on M, that is,
  $D = \ker \omega$ for one-form ω on M with $\omega \wedge \mathrm{d}\omega \neq 0$,
- g smooth fibre inner product on D,

Setting

- M three-dimensional smooth manifold,
- D contact structure on M, that is,
  $D = \ker \omega$ for one-form ω on M with $\omega \wedge \mathrm{d}\omega \neq 0$,
- g smooth fibre inner product on D,
- (D, g) sub-Riemannian structure on M

Setting

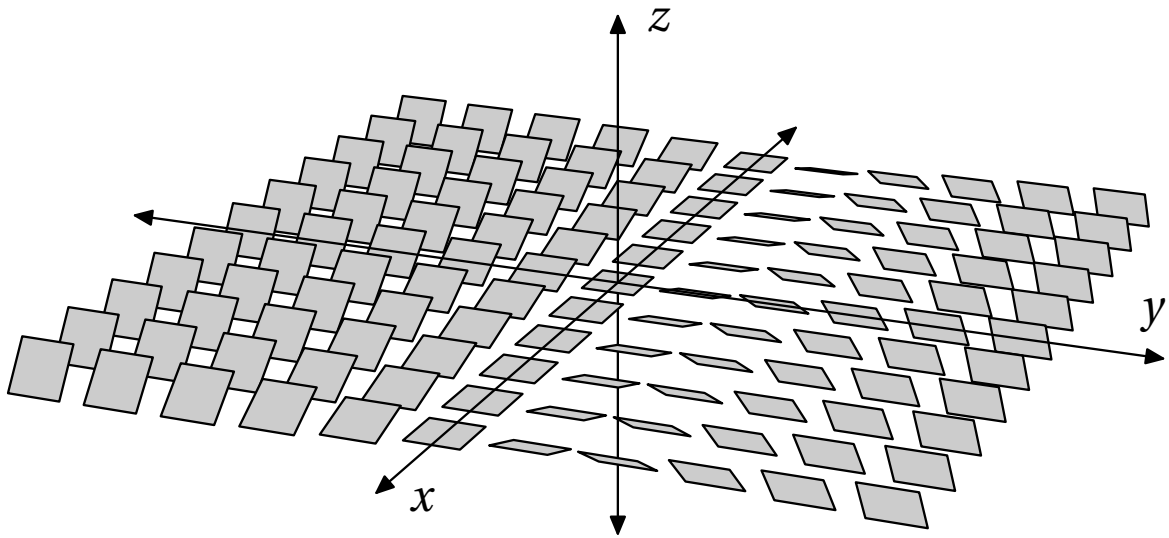
- M three-dimensional smooth manifold,
- D contact structure on M, that is,
  $D = \ker \omega$ for one-form ω on M with $\omega \wedge \mathrm{d}\omega \neq 0$,
- g smooth fibre inner product on D,
- (D, g) sub-Riemannian structure on M



© Wikipedia

Setting

- M three-dimensional smooth manifold,
- D contact structure on M, that is,
  $D = \ker \omega$ for one-form ω on M with $\omega \wedge \mathrm{d}\omega \neq 0$,
- g smooth fibre inner product on D,
- (D, g) sub-Riemannian structure on M,
- S orientable surface embedded in M



© Wikipedia

Adjusted setting (for notational convenience)

- M three-dimensional smooth manifold,

Adjusted setting (for notational convenience)

- M three-dimensional smooth manifold,
- X_1 and X_2 two smooth vector fields on M such that $X_1, X_2, [X_1, X_2]$ are linearly independent everywhere,

Adjusted setting (for notational convenience)

- M three-dimensional smooth manifold,
- X_1 and X_2 two smooth vector fields on M such that $X_1, X_2, [X_1, X_2]$ are linearly independent everywhere,
  - D distribution spanned by X_1 and X_2,

Adjusted setting (for notational convenience)

- M three-dimensional smooth manifold,
- X_1 and X_2 two smooth vector fields on M such that $X_1, X_2, [X_1, X_2]$ are linearly independent everywhere,
  - D distribution spanned by X_1 and X_2,
  - g given by requiring (X_1, X_2) to be an orthonormal frame,

Adjusted setting (for notational convenience)

- M three-dimensional smooth manifold,
- X_1 and X_2 two smooth vector fields on M such that $X_1, X_2, [X_1, X_2]$ are linearly independent everywhere,
  - D distribution spanned by X_1 and X_2,
  - g given by requiring (X_1, X_2) to be an orthonormal frame,
- X_0 the unique vector field on M such that

$$[X_1, X_2] = X_0 + c_{12}^1 X_1 + c_{12}^2 X_2$$

Adjusted setting (for notational convenience)

- M three-dimensional smooth manifold,
- X_1 and X_2 two smooth vector fields on M such that $X_1, X_2, [X_1, X_2]$ are linearly independent everywhere,
  - D distribution spanned by X_1 and X_2,
  - g given by requiring (X_1, X_2) to be an orthonormal frame,
- X_0 the unique vector field on M such that

$$[X_1, X_2] = X_0 + c_{12}^1 X_1 + c_{12}^2 X_2$$

  - X_0 the Reeb vector field for the contact form ω normalised such that $\mathrm{d}\omega|_D = -\operatorname{vol}_g$,

Adjusted setting (for notational convenience)

- M three-dimensional smooth manifold,
- X_1 and X_2 two smooth vector fields on M such that $X_1, X_2, [X_1, X_2]$ are linearly independent everywhere,
  - D distribution spanned by X_1 and X_2,
  - g given by requiring (X_1, X_2) to be an orthonormal frame,
- X_0 the unique vector field on M such that

$$[X_1, X_2] = X_0 + c_{12}^1 X_1 + c_{12}^2 X_2$$

  - X_0 the Reeb vector field for the contact form ω normalised such that $\mathrm{d}\omega|_D = -\operatorname{vol}_g$,
- S embedded surface in M given by

$$S = \{x \in M : u(x) = 0\} \text{ for } u \in C^2(M) \text{ with } \mathrm{d}u \neq 0 \text{ on } S$$

Adjusted setting (for notational convenience)

- M three-dimensional smooth manifold,
- X_1 and X_2 two smooth vector fields on M such that $X_1, X_2, [X_1, X_2]$ are linearly independent everywhere,
  - D distribution spanned by X_1 and X_2,
  - g given by requiring (X_1, X_2) to be an orthonormal frame,
- X_0 the unique vector field on M such that

$$[X_1, X_2] = X_0 + c_{12}^1 X_1 + c_{12}^2 X_2$$

  - X_0 the Reeb vector field for the contact form ω normalised such that $\mathrm{d}\omega|_D = -\operatorname{vol}_g$,
- S embedded surface in M given by

$$S = \{x \in M : u(x) = 0\} \text{ for } u \in C^2(M) \text{ with } \mathrm{d}u \neq 0 \text{ on } S$$

Definition

Set $\Gamma(S)$ of characteristic points on S

$$x \in \Gamma(S) \quad \text{if and only if} \quad (X_1 u)(x) = (X_2 u)(x) = 0$$

Definition

Set $\Gamma(S)$ of characteristic points on S

$$x \in \Gamma(S) \quad \text{if and only if} \quad (X_1 u)(x) = (X_2 u)(x) = 0$$

Construct limiting operator Δ_0 on surface $S \setminus \Gamma(S)$

Definition

Set $\Gamma(S)$ of characteristic points on S

$$x \in \Gamma(S) \quad \text{if and only if} \quad (X_1 u)(x) = (X_2 u)(x) = 0$$

Construct limiting operator Δ_0 on surface $S \setminus \Gamma(S)$

- family of Riemannian manifolds (S, g_ε), for $\varepsilon > 0$, induced by the Riemannian approximations obtained through requiring $(X_1, X_2, \sqrt{\varepsilon} X_0)$ to be a global orthonormal frame,

Definition

Set $\Gamma(S)$ of characteristic points on S

$$x \in \Gamma(S) \quad \text{if and only if} \quad (X_1 u)(x) = (X_2 u)(x) = 0$$

Construct limiting operator Δ_0 on surface $S \setminus \Gamma(S)$

- family of Riemannian manifolds (S, g_ε), for $\varepsilon > 0$, induced by the Riemannian approximations obtained through requiring $(X_1, X_2, \sqrt{\varepsilon} X_0)$ to be a global orthonormal frame,
- family of Laplace–Beltrami operators Δ_ε

Definition

Set $\Gamma(S)$ of characteristic points on S

$$x \in \Gamma(S) \quad \text{if and only if} \quad (X_1 u)(x) = (X_2 u)(x) = 0$$

Construct limiting operator Δ_0 on surface $S \setminus \Gamma(S)$

- family of Riemannian manifolds (S, g_ε), for $\varepsilon > 0$, induced by the Riemannian approximations obtained through requiring $(X_1, X_2, \sqrt{\varepsilon} X_0)$ to be a global orthonormal frame,
- family of Laplace–Beltrami operators Δ_ε

Theorem (Barilari, Boscain, Cannarsa, H)

For $f \in C_c^2(S \setminus \Gamma(S))$, we have

$$\Delta_\varepsilon f \to \Delta_0 f$$

uniformly on $S \setminus \Gamma(S)$ as $\varepsilon \to 0$.

Theorem (Barilari, Boscain, Cannarsa, H)

For $f \in C_c^2(S \setminus \Gamma(S))$, we have

$$\Delta_\varepsilon f \to \Delta_0 f$$

uniformly on $S \setminus \Gamma(S)$ as $\varepsilon \to 0$.

Theorem (Barilari, Boscain, Cannarsa, H)

For $f \in C_c^2(S \setminus \Gamma(S))$, we have

$$\Delta_\varepsilon f \to \Delta_0 f$$

uniformly on $S \setminus \Gamma(S)$ as $\varepsilon \to 0$.

The limiting operator is given by

$$\Delta_0 = \widehat{X}_S^2 + b\widehat{X}_S \ ,$$

Theorem (Barilari, Boscain, Cannarsa, H)

For $f \in C_c^2(S \setminus \Gamma(S))$, we have

$$\Delta_\varepsilon f \to \Delta_0 f$$

uniformly on $S \setminus \Gamma(S)$ as $\varepsilon \to 0$.

The limiting operator is given by

$$\Delta_0 = \widehat{X}_S^2 + b\widehat{X}_S \ ,$$

where

$$\widehat{X}_S = \frac{(X_2 u)X_1 - (X_1 u)X_2}{\sqrt{(X_1 u)^2 + (X_2 u)^2}}$$

Theorem (Barilari, Boscain, Cannarsa, H)

For $f \in C_c^2(S \setminus \Gamma(S))$, we have

$$\Delta_\varepsilon f \to \Delta_0 f$$

uniformly on $S \setminus \Gamma(S)$ as $\varepsilon \to 0$.

The limiting operator is given by

$$\Delta_0 = \widehat{X}_S^2 + b\widehat{X}_S \ ,$$

where

$$\widehat{X}_S = \frac{(X_2 u)X_1 - (X_1 u)X_2}{\sqrt{(X_1 u)^2 + (X_2 u)^2}} \quad \text{and} \quad b = \frac{X_0 u}{\sqrt{(X_1 u)^2 + (X_2 u)^2}} \ .$$

The limiting operator is given by

$$\Delta_0 = \widehat{X}_S^2 + b\widehat{X}_S \ ,$$

where

$$\widehat{X}_S = \frac{(X_2u)X_1 - (X_1u)X_2}{\sqrt{(X_1u)^2 + (X_2u)^2}} \quad \text{and} \quad b = \frac{X_0u}{\sqrt{(X_1u)^2 + (X_2u)^2}} \ .$$

The limiting operator is given by

$$\Delta_0 = \widehat{X}_S^2 + b\widehat{X}_S \ ,$$

where

$$\widehat{X}_S = \frac{(X_2u)X_1 - (X_1u)X_2}{\sqrt{(X_1u)^2 + (X_2u)^2}} \quad \text{and} \quad b = \frac{X_0u}{\sqrt{(X_1u)^2 + (X_2u)^2}} \ .$$

Proposition (Barilari, Boscain, Cannarsa, H)

Let K_ε be the Gaussian curvature of the Riemannian manifold (S, g_ε). We have

$$K_0 := \lim_{\varepsilon \to 0} K_\varepsilon = -\widehat{X}_S(b) - b^2$$

uniformly on compact subsets of $S \setminus \Gamma(S)$.

The limiting operator is given by

$$\Delta_0 = \widehat{X}_S^2 + b\widehat{X}_S \ ,$$

where

$$\widehat{X}_S = \frac{(X_2u)X_1 - (X_1u)X_2}{\sqrt{(X_1u)^2 + (X_2u)^2}} \quad \text{and} \quad b = \frac{X_0u}{\sqrt{(X_1u)^2 + (X_2u)^2}} \ .$$

Proposition (Barilari, Boscain, Cannarsa, H)

Let K_ε be the Gaussian curvature of the Riemannian manifold (S, g_ε). We have

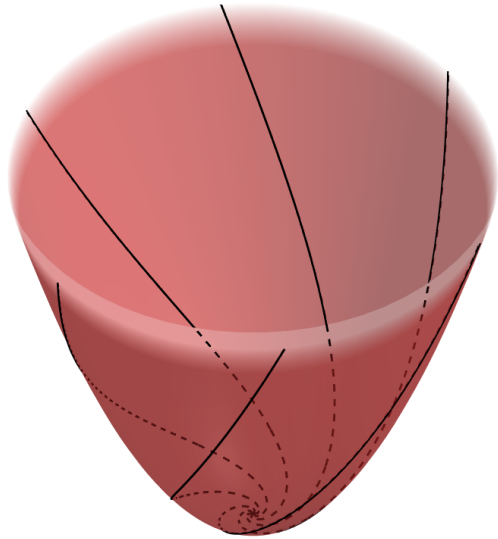
$$K_0 := \lim_{\varepsilon \to 0} K_\varepsilon = -\widehat{X}_S(b) - b^2$$

uniformly on compact subsets of $S \setminus \Gamma(S)$.

Notion of intrinsic Gaussian curvature K_0 for the surface S

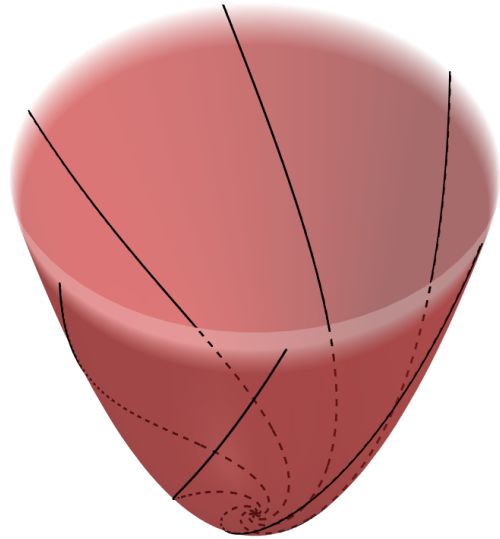
Example: Paraboloid $z = a\left(x^2 + y^2\right)$ in the Heisenberg group

Example: Paraboloid $z = a\left(x^2 + y^2\right)$ in the Heisenberg group



Characteristic foliation described by logarithmic spirals

Example: Paraboloid $z = a\left(x^2 + y^2\right)$ in the Heisenberg group

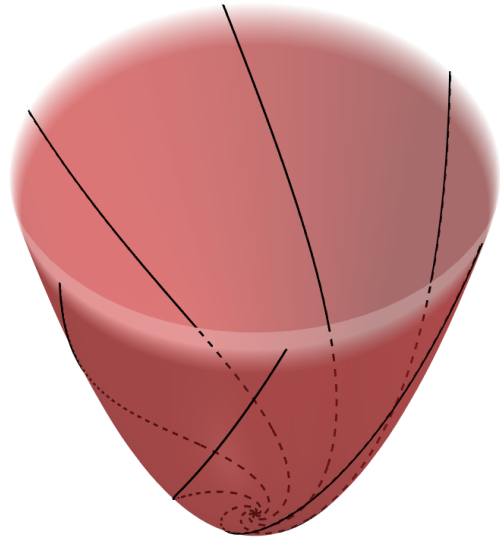


Characteristic foliation described by logarithmic spirals

In coordinates (s, ψ) with $s > 0$ and $\psi \in [0, 2\pi)$ on $S \setminus \Gamma(S)$

$$\widehat{X}_S = \frac{\partial}{\partial s} \quad \text{and} \quad b(s, \psi) = \frac{2}{s} \,,$$

Example: Paraboloid $z = a\left(x^2 + y^2\right)$ in the Heisenberg group



Characteristic foliation described by logarithmic spirals

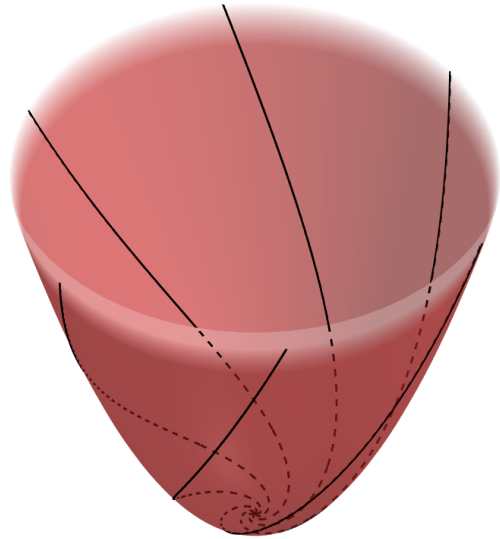
In coordinates (s, ψ) with $s > 0$ and $\psi \in [0, 2\pi)$ on $S \setminus \Gamma(S)$

$$\widehat{X}_S = \frac{\partial}{\partial s} \quad \text{and} \quad b(s, \psi) = \frac{2}{s} \,,$$

so

$$\Delta_0 = \quad \frac{\partial^2}{\partial s^2} + \frac{2}{s}\frac{\partial}{\partial s} \,.$$

Example: Paraboloid $z = a\left(x^2 + y^2\right)$ in the Heisenberg group



Characteristic foliation described by logarithmic spirals

In coordinates (s, ψ) with $s > 0$ and $\psi \in [0, 2\pi)$ on $S \setminus \Gamma(S)$

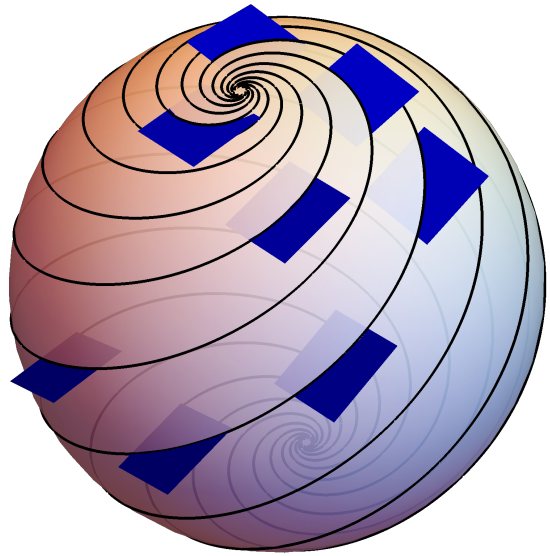
$$\widehat{X}_S = \frac{\partial}{\partial s} \quad \text{and} \quad b(s, \psi) = \frac{2}{s} \,,$$

so

$$\frac{1}{2}\Delta_0 = \frac{1}{2}\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} \,.$$

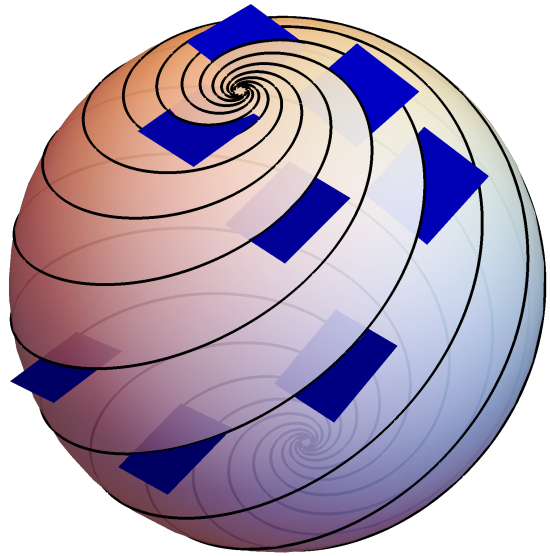
Example: Sphere $x^2 + y^2 + z^2 = a^2$ in the Heisenberg group

Example: Sphere $x^2 + y^2 + z^2 = a^2$ in the Heisenberg group



Characteristic foliation described by loxodromes

Example: Sphere $x^2 + y^2 + z^2 = a^2$ in the Heisenberg group



Characteristic foliation described by loxodromes

In coordinates (s, φ) on $S \setminus \Gamma(S)$

$$\frac{1}{2}\Delta_0 = \frac{1}{2}\frac{\partial^2}{\partial s^2} + \left(\cot\left(\theta(s)\right)\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)\frac{\partial}{\partial s}\,,$$

where $\varphi \in [0, 2\pi)$ and s is given in terms of the polar angle θ as a multiple of an elliptic integral of the second kind.

We can recover three classes of familiar stochastic processes.

We can recover three classes of familiar stochastic processes.

- Bessel process of order 3 has generator

$$\frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \quad \text{for } r > 0$$

We can recover three classes of familiar stochastic processes.

- Bessel process of order 3 has generator

$$\frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \quad \text{for } r > 0$$

- Legendre process of order 3 has generator

$$\frac{1}{2}\frac{\partial^2}{\partial \theta^2} + k\cot(k\theta)\frac{\partial}{\partial \theta} \quad \text{for } \theta \in \left(0, \frac{\pi}{k}\right)$$

We can recover three classes of familiar stochastic processes.

- Bessel process of order 3 has generator

$$\frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \quad \text{for } r > 0$$

- Legendre process of order 3 has generator

$$\frac{1}{2}\frac{\partial^2}{\partial \theta^2} + k\cot(k\theta)\frac{\partial}{\partial \theta} \quad \text{for } \theta \in \left(0, \frac{\pi}{k}\right)$$

- hyperbolic Bessel process of order 3 has generator

$$\frac{1}{2}\frac{\partial^2}{\partial r^2} + k\coth(kr)\frac{\partial}{\partial r} \quad \text{for } r > 0$$

Criterion for accessibility of characteristic points

Criterion for accessibility of characteristic points in terms of

$$\operatorname{Hess} u = \begin{pmatrix} X_1 X_1 u & X_1 X_2 u \\ X_2 X_1 u & X_2 X_2 u \end{pmatrix}$$

Criterion for accessibility of characteristic points in terms of

$$\operatorname{Hess} u = \begin{pmatrix} X_1X_1u & X_1X_2u \\ X_2X_1u & X_2X_2u \end{pmatrix}$$

Definition

Characteristic point $x \in \Gamma(S)$ is called

elliptic	if $\det((\operatorname{Hess} u)(x)) > 0$,
hyperbolic	if $\det((\operatorname{Hess} u)(x)) < 0$.

Criterion for accessibility of characteristic points in terms of

$$\operatorname{Hess} u = \begin{pmatrix} X_1X_1u & X_1X_2u \\ X_2X_1u & X_2X_2u \end{pmatrix}$$

Definition

Characteristic point $x \in \Gamma(S)$ is called

elliptic	if $\det((\operatorname{Hess} u)(x)) > 0$,
hyperbolic	if $\det((\operatorname{Hess} u)(x)) < 0$.

Theorem (Barilari, Boscain, Cannarsa, H)

For the canonical stochastic process with generator $\frac{1}{2}\Delta_0$

- elliptic characteristic points are inaccessible, while
- hyperbolic characteristic points are accessible from the separatrices.

Definition

Characteristic point $x \in \Gamma(S)$ is called

$$\begin{array}{ll} \text{elliptic} & \text{if } \det((\operatorname{Hess} u)(x)) > 0\,, \\ \text{hyperbolic} & \text{if } \det((\operatorname{Hess} u)(x)) < 0\,. \end{array}$$

Theorem (Barilari, Boscain, Cannarsa, H)

For the canonical stochastic process with generator $\frac{1}{2}\Delta_0$

- elliptic characteristic points are inaccessible, while
- hyperbolic characteristic points are accessible from the separatrices.

Definition

Characteristic point $x \in \Gamma(S)$ is called

$$\begin{array}{ll} \text{elliptic} & \text{if } \det((\operatorname{Hess} u)(x)) > 0\,, \\ \text{hyperbolic} & \text{if } \det((\operatorname{Hess} u)(x)) < 0\,. \end{array}$$

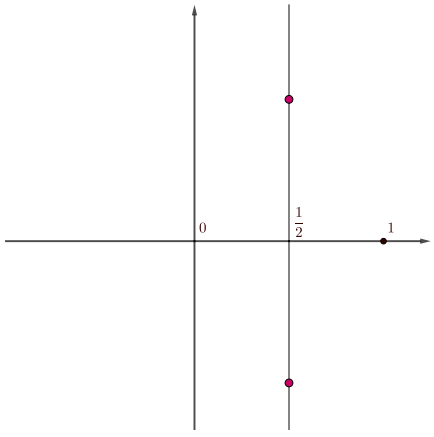
Theorem (Barilari, Boscain, Cannarsa, H)

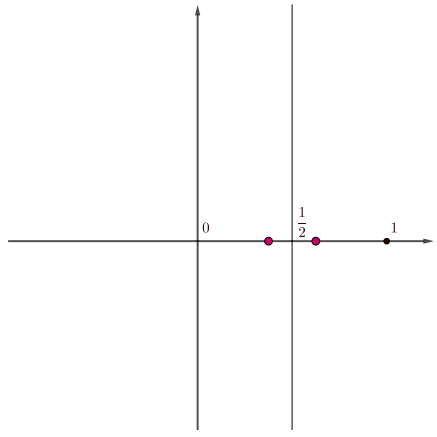
For the canonical stochastic process with generator $\frac{1}{2}\Delta_0$

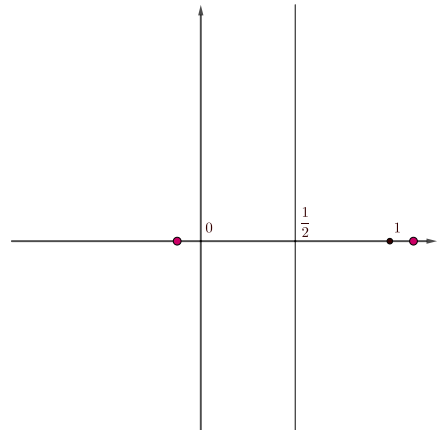
- elliptic characteristic points are inaccessible, while
- hyperbolic characteristic points are accessible from the separatrices.

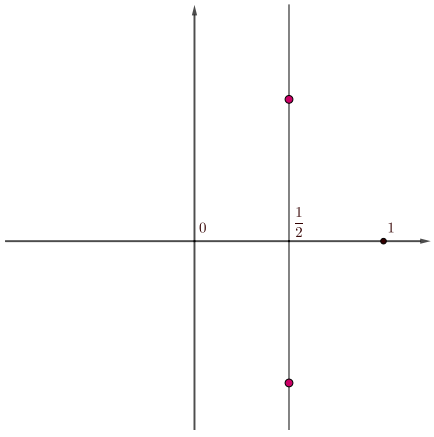
Proof uses the eigenvalues of

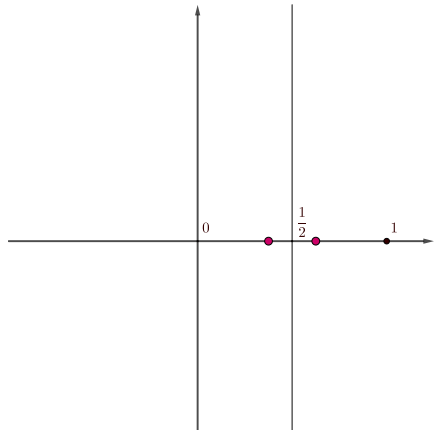
$$\begin{pmatrix} X_1X_2u & -X_1X_1u \\ X_2X_2u & -X_2X_1u \end{pmatrix}(x) \quad \text{subject to} \quad (X_0u)(x) = 1\,.$$

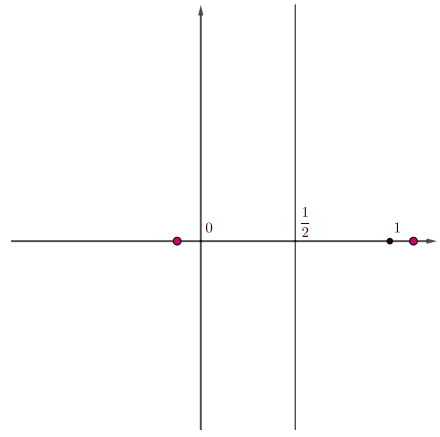

0
1/2
1

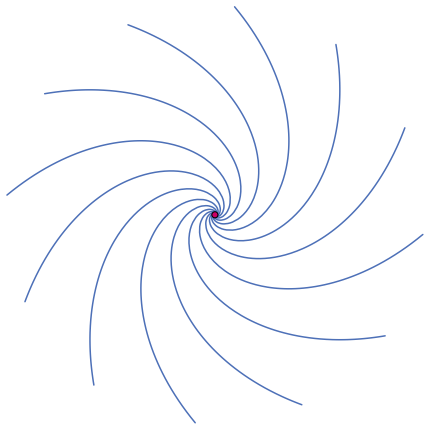

0
1/2
1

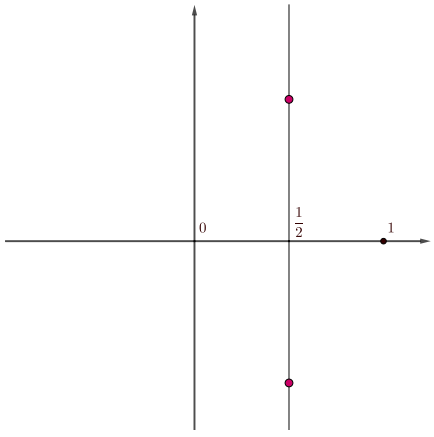

0
1/2
1

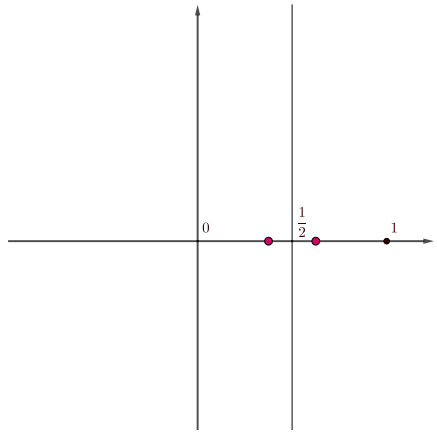

0
1/2
1

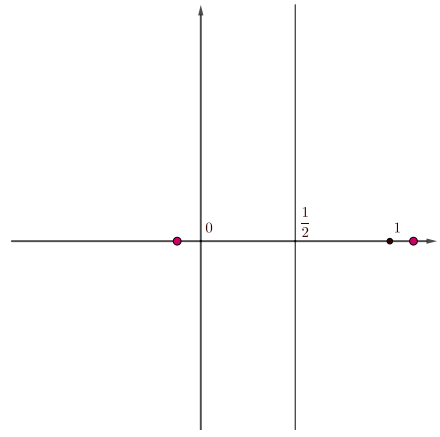

0
1/2
1

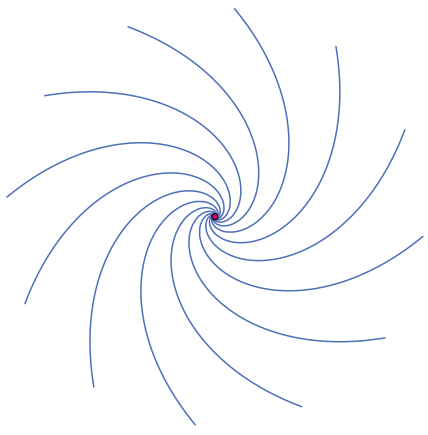

0
1/2
1

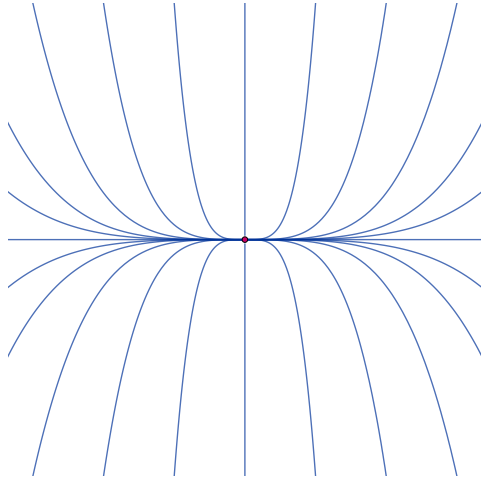


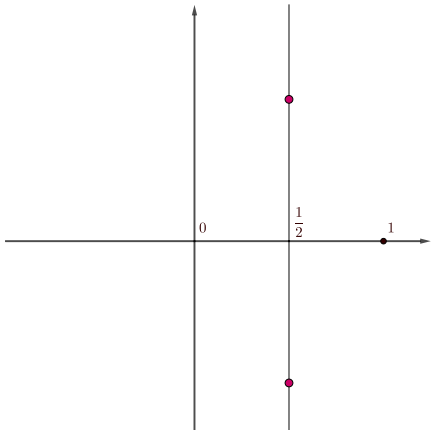

0
1/2
1

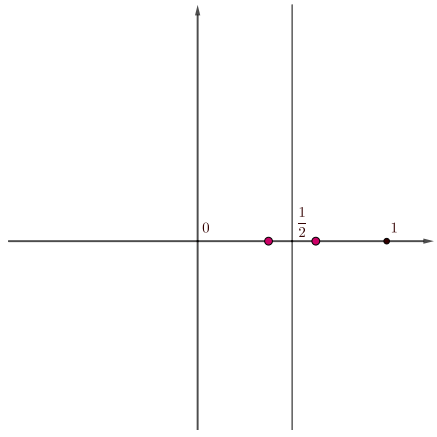

0
1/2
1

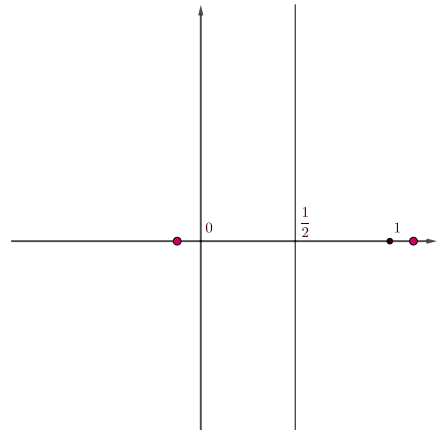

0
1/2
1

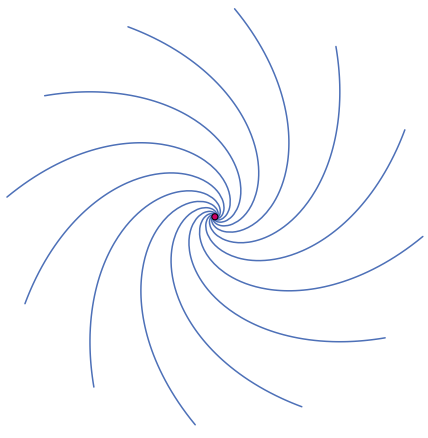


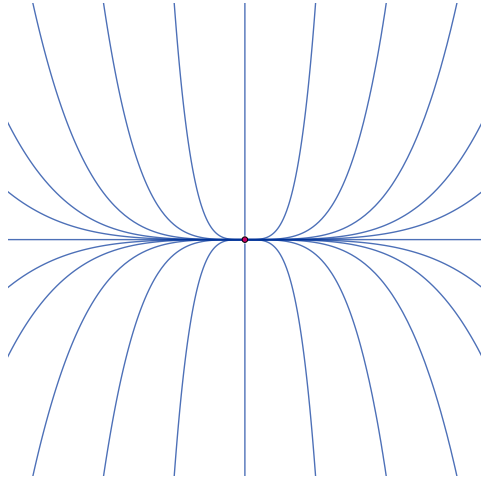


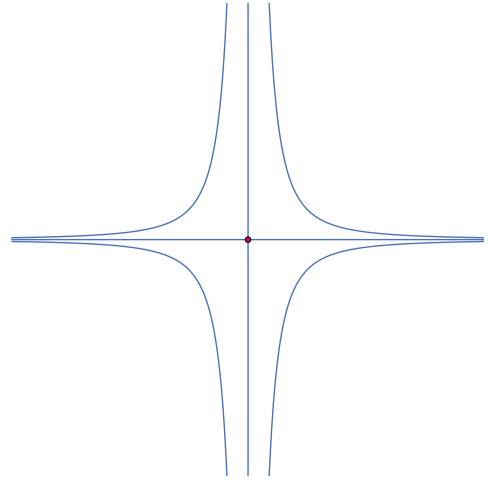

0
1/2
1

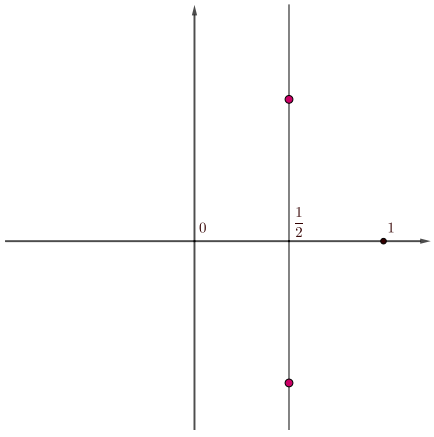

0
1/2
1

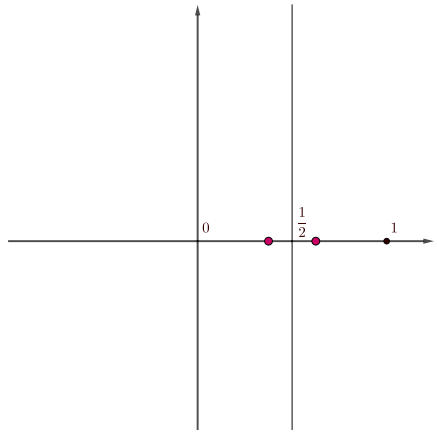

0
1/2
1

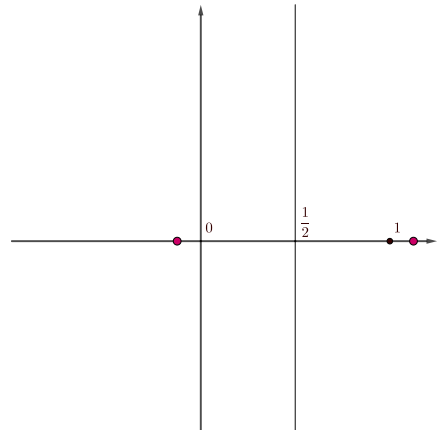


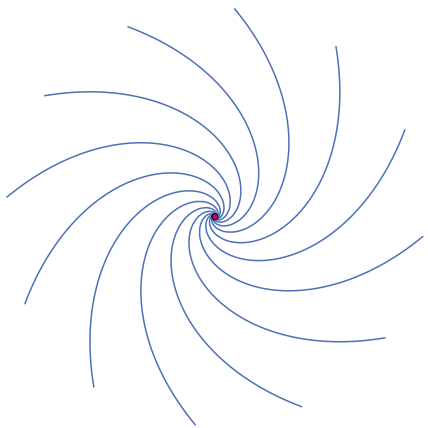


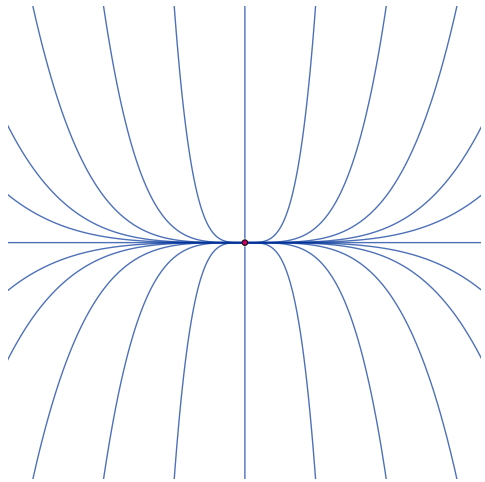


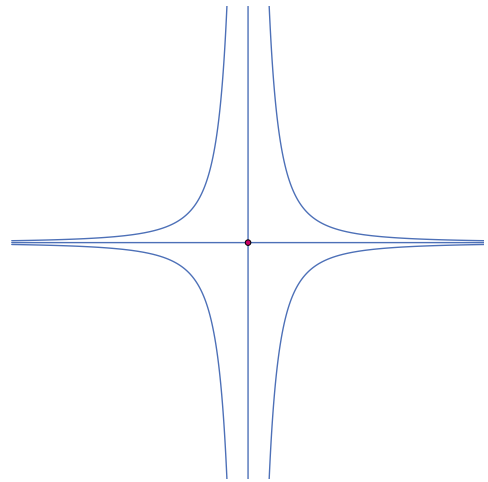




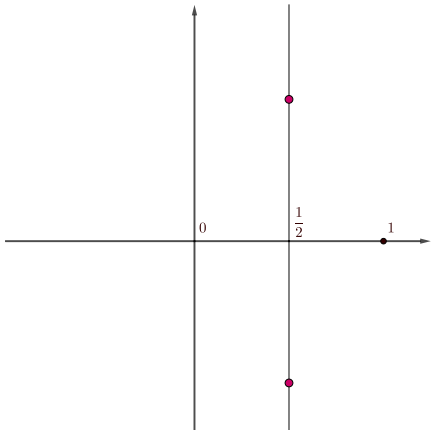


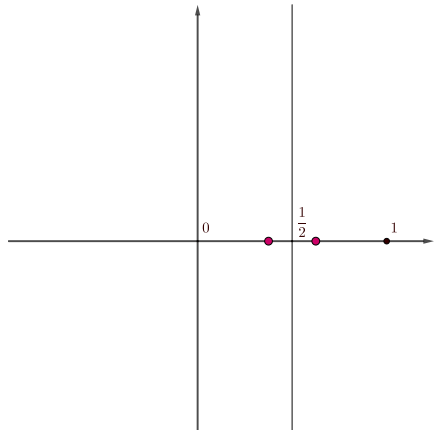


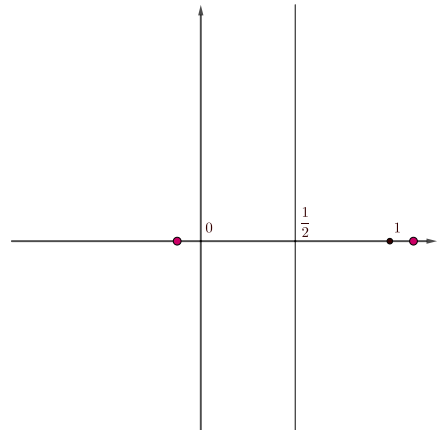


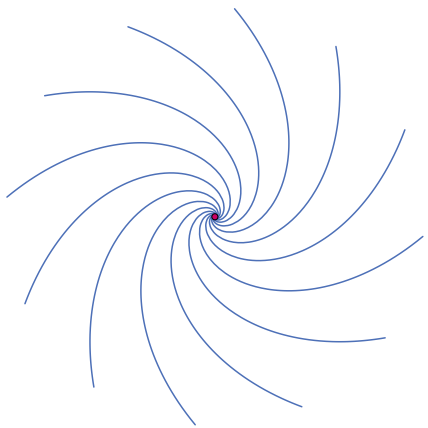


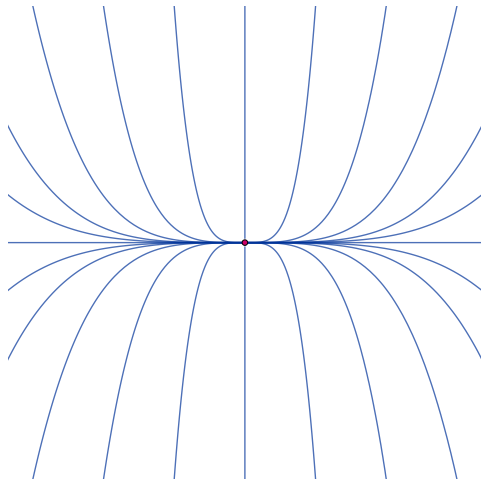
$$b(\gamma(s)) \sim \frac{2}{s}$$

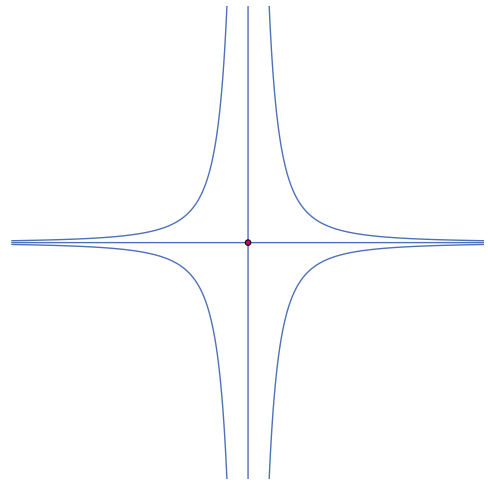












$$b(\gamma(s)) \sim \frac{2}{s}$$

$$b(\gamma(s)) \sim \frac{1}{\lambda_i s}$$

$$b(\gamma(s)) \sim \frac{1}{\lambda_i s}$$