On the Pontryagin Maximum Principle

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Outline

Example 1: Fishing problem *[Clark, 1974]*

State variable: x size of the fish population

Control variable: u fishing effort

Cost: net revenue in a fix time-interval $[0, T]$

$$
\begin{aligned}
\max \quad & \int_0^T \left(E u(t) - \frac{c}{x(t)} u(t) \right) \mathrm{d}t, \\
\text{s.t.} \quad & \dot{x}(t) = r\, x(t) \left(1 - x(t)/k\right) - u(t), \\
& 0 \le u(t) \le U_{\max}, \quad \text{a.e.} \quad t \in [0, T], \\
& x(t) \ge 0 \quad \text{on } [0, T], \\
& x(0) = x_0, \quad x(T) \ \text{free}.
\end{aligned}
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r : difference between reproduction and mortality rate for fish

$\frac{rx}{k}$: mortality rate given by resources competition

E : sale price, $\frac{c}{x}$: fishing cost,

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Example 2: Goddard problem in 1D *[Goddard, 1919] [Seywald & Cliff, 1993]*

State variables: (h, v, m) altitude, speed and mass

Control variable: u thrust

Goal: to minimize fuel consumption

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\max\ & m(T), \\
\text{s.t.}\ & \dot{h}(t) = v(t), \\
& \dot{v}(t) = -1/h(t)^2 + 1/m(t)\Big(u(t) - D(h(t), v(t))\Big), \\
& \dot{m}(t) = -b\,u(t), \\
& 0 \le u(t) \le U_{\max}, \quad \text{a.e.} \quad t \in [0, T], \\
& h(0) = 0,\ v(0) = 0,\ m(0) = 1,\ h(T) = 1,
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T : free final time, b : fuel consumption coefficient,

U_{max} : maximum thrust, $D(h, v)$: atmospheric drag.

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$U_{\max}$: maximum thrust, $D(h, v)$: atmospheric drag.

Formulation of a *standard* optimal control problem

$$
\begin{aligned}
\max \quad & \int_0^T L(t, x(t), u(t))\mathrm{d}t + \Psi(T, x(T)), \\
\text{s.t.} \quad & \dot{x}(t) = f(t, x(t), u(t)), \quad \text{a.e. } t \in [0, T], \\
& x(0) = x_0, \\
& (T, x(T)) \in \mathcal{T} \subseteq \mathbb{R}^{n+1}, \\
& u(t) \in U \subseteq \mathbb{R}^m, \quad \text{a.e. } t \in [0, T],
\end{aligned}
$$

with

- $u \in L^1([0, T]; U)$,
- $x \in AC([0, T]; \mathbb{R}^n)$,
- U is, in general, a compact set of $\mathbb{R}^m$.

Remark: reduction of the problem to one with terminal cost

Bolza form:

$$\int_0^T L(t, x(t), u(t)) \mathrm{d}t + \Psi(T, x(T)).$$

Lagrange form:

$$\int_0^T L(t, x(t), u(t)) \mathrm{d}t.$$

Mayer form:

$$\Psi(T, x(T)).$$

From Bolza or Lagrange to Mayer form: consider the additional state variable

$$\dot{x}_{n+1}(t) = L(t, x(t), u(t)),$$

and the terminal cost

$$x_{n+1}(T) + \Psi(T, x(T)).$$

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We simplify a bit our problem...

...we **remove the final constraints,** leave only terminal cost.

Let the problem be written in its Mayer form:

$$
\text{(P)} \qquad \begin{aligned} \max \ & \Psi(x(T)), \\ \text{s.t.} \ & \dot{x}(t) = f(t, x(t), u(t)), \\ & x(0) = x_0, \\ & u(t) \in U, \end{aligned}
$$

with $u \in L^1([0,T];U)$, $x \in AC([0,T];I\!R^n)$.

Recalling *Lagrange multipliers*

Let us recall a very simple problem from Calculus:

$$\begin{aligned} \max \;\; & f(x), \\ \text{s.a.} \;\; & g(x) = 0, \end{aligned}$$

where $f, g \colon \mathbb{R}^n \to \mathbb{R}$.

Let us define the **Lagrangian**:

$$\mathcal{L}(x, \lambda) := f(x) + \lambda g(x), \qquad \lambda \in \mathbb{R}^{n,*}.$$

Lagrange multipliers' Method (an optimality condition):

if x_0 is optimal then there exists a multiplier $\lambda_0 \in \mathbb{R}^{n,*}$ such that

$$\begin{aligned} D_x \mathcal{L}(x_0, \lambda_0) = 0, \\ D_\lambda \mathcal{L}(x_0, \lambda_0) = 0. \end{aligned}$$

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Formally applying the Lagrange multipliers' Method...

The Lagrangian is

$$\mathcal{L}(x,u,p) := \Psi(x(T)) + \int_0^T p(t)\big(f(t,x(t),u(t)) - \dot{x}(t)\big)\mathrm{d}t,$$

where p is a multiplier. Then

$$D_x\mathcal{L}(x,u,p)z = \nabla\Psi(x(T))z(T) + \int_0^T p(t)\big(D_x f(t,x(t),u(t))z(t) - \dot{z}(t)\big)\mathrm{d}t.$$

Integrating by parts:

$$D_x\mathcal{L}(x,u,p)z = \Big(\underbrace{\nabla\Psi(x(T)) - p(T)}_{\text{final condition for } p}\Big)z(T) + p(0)\underbrace{z_0}_{=0} + \int_0^T \Big(\underbrace{p(t)D_x f(t,x(t),u(t)) + \dot{p}(t)}_{\text{dynamics for } p}\Big)z(t)\mathrm{d}t.$$

Pontryagin's Maximum Principle (PMP) with no final constraints

Local optimality: optimality in an L^1-neighbourhood of the control

Theorem

Let u^* *be an* optimal control *for* (P), x^* *the associated trajectory,* $p : [0,T] \to \mathbb{R}^{n,*}$ *the solution of the adjoint equation:*

$$\dot{p}(t) = -p(t) D_x f(t, x^*(t), u^*(t)),$$

with the transversality condition:

$$p(T) = \nabla \Psi(x^*(T)).$$

Then, the following maximum condition holds:

$$p(t) f(t, x^*(t), u^*(t)) = \max_{\omega \in U} p(t) f(t, x^*(t), \omega), \quad \textit{a.e. } t \in [0,T].$$

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Another way of writing Pontryagin's Maximum Principle

The **Hamiltonian:** (or pre-Hamiltonian, or *unmaximized* Hamiltonian)

$$H(t, x, u, p) := pf(t, x, u).$$

Pontryagin's Principle says that:

"If (x^*, u^*) is an L^1 local minimum and $p : [0, T] \to \mathbb{R}^{n,*}$ is the solution of the adjoint equation with the transversality condition:

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Proof

We have to prove that: L^1-local optimality of u^* $\Rightarrow$ Maximum condition

Hypothesis: L^1-local optimality of u^*.

Given $\{(x_\varepsilon, u_\varepsilon)\}_{\varepsilon>0}$ a family of trajectory-control pairs with $u_\varepsilon \xrightarrow[\varepsilon\to 0+]{L^1} u^*$, we have

$$0 \geq \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0+} \Psi(x_\varepsilon(T)) = \nabla\Psi(x^*(T)) \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0+} x_\varepsilon(T) = p(T) \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0+} x_\varepsilon(T).$$

Thesis: the solution p of the adjoint equation with the transversality condition satisfies the maximum condition.

Thus, we want to prove that

$$0 \geq p(\tau)\Big(f(\tau, x^*(\tau), \omega) - f(\tau, x^*(\tau), u^*(\tau))\Big), \quad \text{a.e. on } [0,T], \quad \text{for all } \omega \in U.$$

We have information at the endpoint and we need it at a.e. point of $[0,T]$.

Elements of the proof:

- The adjoint and the variational equations.
- Perturbations of the optimal control: known as *needle variations*.

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Proof (continuation): the variational equation

For $\mathbf{y} \in I\!R^n$, let $v : [\tau, T] \to I\!R^m$ be the solution of the variational equation

$$\dot{v}(t) = D_x f(t, x^*(t), u^*(t))\, v(t), \quad v(\tau) = \mathbf{y}.$$

Take $x_\varepsilon(\tau) = x^*(\tau) + \varepsilon\mathbf{y} + o(\varepsilon)$. Then, for all $t \geq \tau$,

$$v(t) = \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0^+} x_\varepsilon(t).$$

This is, $\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0^+} x_\varepsilon(\tau) = \mathbf{y} \implies v(t) = \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0^+} x_\varepsilon(t)$

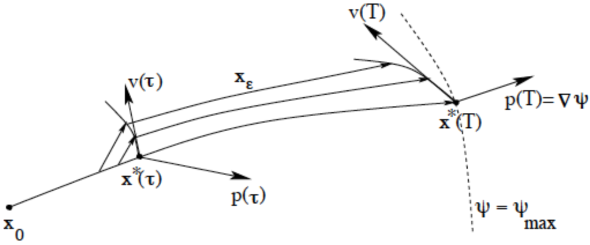

Figure: Figure from Bressan-Piccoli's book

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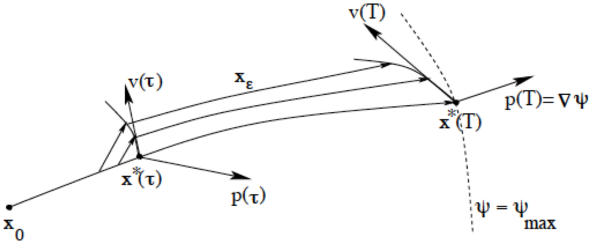

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Proof (continuation)

If v and p are solutions of

$$\begin{cases} \dot{v}(t) = D_x f(t, x^*(t), u^*(t))\, v(t), \\ \dot{p}(t) = -p(t)\, D_x f(t, x^*(t), u^*(t)), \end{cases}$$

then

$$t \mapsto p(t) \cdot v(t) \text{ is constant.}$$

Recall: we have

$$0 \geq p(T) \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon = 0+} x_\varepsilon(T),$$

and we want

$$0 \geq p(\tau) \Big(f(\tau, x^*(\tau), \omega) - f(\tau, x^*(\tau), u^*(\tau)) \Big).$$

To conclude the proof we construct $\{(x_\varepsilon, u_\varepsilon\}_\varepsilon$ such that

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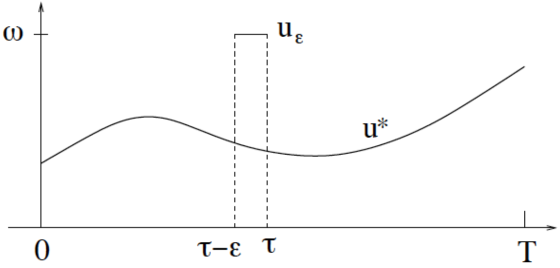
Proof (continuation): needle variations

For any $\tau \in (0, T]$, $\omega \in U$, $0 < \varepsilon < \tau$, we define the needle variations

$$u_\varepsilon(t) := \begin{cases} \omega, & \text{if } t \in [\tau - \varepsilon, \tau], \\ u^*(t), & \text{if not.} \end{cases}$$



Claim

$$\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0^+} x_\varepsilon(\tau) = f(\tau, x^*(\tau), \omega) - f(\tau, x^*(\tau), u^*(\tau)), \quad \text{for a.e. } \tau \in (0, T].$$

■

Proof (addendum): idea for proof of the Claim

We have,

$$\frac{x_\varepsilon(\tau) - x^*(\tau)}{\varepsilon} = \frac{1}{\varepsilon}\left\{\int_{\tau-\varepsilon}^{\tau} f(t, x_\varepsilon(t), \omega)\mathrm{d}t - \int_{\tau-\varepsilon}^{\tau} f(t, x^*(t), u^*(t))\mathrm{d}t\right\}.$$

Observe that

$$\frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau} f(t, x^*(t), u^*(t))\mathrm{d}t \underset{\varepsilon\to 0+}{\longrightarrow} f(\tau, x^*(\tau), u^*(\tau)),$$

if τ is a *Lebesgue point* of $t \mapsto f(t, x^*(t), u^*(t))$, and

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau} \big(f(t, x_\varepsilon(t), \omega) - f(\tau, x_\varepsilon(\tau), \omega)\big)\mathrm{d}t \\
&\qquad = \frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau} \Big(f(t, x_\varepsilon(t), \omega) - f(t, x_\varepsilon(\tau), \omega) + f(t, x_\varepsilon(\tau), \omega) - f(\tau, x_\varepsilon(t), \omega)\Big)\mathrm{d}t \\
&\qquad \leq \frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau} \Big(L|x_\varepsilon(t) - x_\varepsilon(\tau)| + f(t, x_\varepsilon(\tau), \omega) - f(\tau, x_\varepsilon(\tau), \omega)\Big)\mathrm{d}t \underset{\varepsilon\to 0+}{\longrightarrow} 0,
\end{aligned}$$

if τ is a *Lebesgue point* of $t \mapsto f(t, x^*(\tau), \omega)$.

$\square$

Summary of the proof

- We generate variations of the control modifying it around a time τ.
- We transport the tangent vector generated by the variations from $t = \tau$ to $t = T$, by means of the variational equation.
- We use the constancy of $t \mapsto p(t) \cdot v(t)$ to show that its sign at $t = T$ is maintained along $[0, T]$.

A simple illustrative example

$$
\begin{aligned}
\max \ & x_1(T), \\
\text{s.t. } & \dot{x}_1(t) = x_2(t), \\
& \dot{x}_2(t) = -x_1(t) + u(t), \\
& x(0) = (0, 0), \\
& -1 \le u(t) \le 1.
\end{aligned}
$$

The adjoint equation and the transversality condition give

$$
\dot{p}_1 = p_2, \quad \dot{p}_2 = -p_1,
$$
$$
p_1(T) = 1, \quad p_2(T) = 0.
$$

Then

$$
p_1(t) = \cos(T - t), \quad p_2(T) = \sin(T - t),
$$

and the optimal control u^* satisfies

$$
p_1(t)x_2(t) + p_2(t)(-x_1(t) + u^*(t)) = \max_{\omega \in [-1,1]} \{p_1 x_2 - p_2 x_1 + p_2 \omega\}.
$$

$$
u^*(t) = \text{sign}(p_2(t)) = \text{sign}(\sin(T - t)).
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□

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$$

$$
u^*(t) = \operatorname{sign}(p_2(t)) = \operatorname{sign}(\sin(T - t)).
$$

$\square$

Example for production and reinvestment in a factory

x : profit per unit time

u : percentage to be reinvested

Given $T > 0$, $x_0 \in I\!R$, let us consider the optimal control problem:

$$
\begin{aligned}
\max \int_0^T & \big(x(t) - u(t)x(t)\big)\mathrm{d}t, \\
& \dot{x}(t) = ku(t)x(t), \\
& x(0) = x_0, \\
& 0 \le u(t) \le 1.
\end{aligned}
$$

Here $k > 0$ is the revenue rate for the reinvestment.

Factory example (continuation)

We choose $T > 1$, $x_0 > 0$, $k = 1$.
We add the state variable $\dot{y} = x - ux$, with $y(0) = 0$, and change the integral cost by the terminal cost

$$y(T).$$

The Hamiltonian is

$$H(x, y, u, p, q) = pux + (1 - u)x = x + ux(p - 1),$$

Then q has $\dot{q} = 0$ and $q(T) = 1$. This is, $q \equiv 1$.

For p one has

(ADJ) $$\dot{p}(t) = -D_x H = -1 - u(t)(p(t) - 1),$$

(T) $$p(T) = \nabla\phi(x(T)) = 0,$$

and the following maximum condition is satisfied:

(M) $$H(x(t), u(t), p(t)) = \max_{0 \le v \le 1} \big\{ x(t) + vx(t)(p(t) - 1) \big\}.$$

If $p(t) = 1$ in a positive measure interval, then necessarily $\dot{p}(t) = 0$ on that interval, but from (ADJ) we get $\dot{p}(t) = -1$. **Contradiction!**

Factory example (continuation)

Note that $x = 0$ is an equilibrium of the system and that $x(0) = x_0 > 0$, then there is $\tilde{T}$ such that

$$x(t) > 0, \quad t \in [0, \tilde{T}).$$

If $x(\tilde{T}) = 0$ and $\tilde{T} < T$, then $x = 0$ on $[\tilde{T}, T]$.
Observe as well that $y(\tilde{T}) = y(T)$, so that we can solve the problem in the interval $[0, \tilde{T}]$.

Note that (M) implies:

$$u(t) = \begin{cases} 1 & \text{if } p(t) > 1, \\ 0 & \text{if } p(t) < 1. \end{cases}$$

Given that $p(\tilde{T}) = 0$, then $p \leq 1$ near $\tilde{T}$, $t < \tilde{T}$. Then, $u = 0$ near $\tilde{T}$. Thus $\dot{p}(t) = -1$, and

$$p(t) = \tilde{T} - t, \qquad \text{for } \tilde{T} - 1 \leq t \leq \tilde{T}.$$

For $t = \tilde{T} - 1$, we have $p(t) = 1$, $\dot{p}(t) = -1 < 0$. Then p is decreasing, thus $p > 1$ for $t < \tilde{T}$, and $u = 1$ and $\dot{p} = -p$. We get

$$u^*(t) = \begin{cases} 1 & [0, \tilde{T} - 1), \\ 0 & [\tilde{T} - 1, \tilde{T}] \cup (\tilde{T}, T]. \end{cases}$$

■

Finite-horizon linear quadratic regulator (LQR)

Let us consider the problem

$$\min \int_{t_0}^{t_1} \Big(x^\top(t)Q(t)x(t) + u^\top(t)R(t)u(t)\Big)dt + x^\top(t_1)Mx(t_1)$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$

$$x(t_0) = x_0.$$

Here M and $Q(t)$ are symmetric positive semidefinite, and $R(t)$ is symmetric positive definite on $[0, T]$.

The Hamiltonian is (using column adjoint state for simplicity $p^\top \mapsto p$):

$$H(t, x, u, p) = p^\top A(t)x + p^\top B(t)u - x^\top Q(t)x - u^\top R(t)u.$$

Here u is unconstrained so we have

$$u^*(t) = \tfrac{1}{2}R^{-1}(t)B^\top(t)p(t).$$

This is the unique control verifying the PMP. *It remains to compute x and p.*

Liberzon, D.,
Calculus of variations and optimal control theory: a concise introduction.
Princeton University Press, 2011

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LQR (continuation)

For the adjoint variable we have

$$\dot{p} = 2Q(t)x^*(t) - A^\top(t)p(t), \quad p(t_1) = -2Mx^*(t_1),$$

and we get the following system for x^* and p :

$$\begin{pmatrix} \dot{x}^* \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A(t) & \frac{1}{2}B(t)R^{-1}(t)B^\top(t) \\ 2Q(t) & -A^\top(t) \end{pmatrix} \begin{pmatrix} x^* \\ p \end{pmatrix}$$

If Φ is the transition matrix of this system, *i.e.*, $\begin{pmatrix} x^*(t) \\ p(t) \end{pmatrix} = \Phi(t,s) \begin{pmatrix} x^*(s) \\ p(s) \end{pmatrix}$ then,

$$\begin{pmatrix} x^*(t) \\ p(t) \end{pmatrix} = \Phi(t,t_1) \begin{pmatrix} x^*(t_1) \\ p(t_1) \end{pmatrix}.$$

Partitioning into blocks:

$$\begin{pmatrix} x^*(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \Phi_{11}(t,t_1) & \Phi_{12}(t,t_1) \\ \Phi_{21}(t,t_1) & \Phi_{22}(t,t_1) \end{pmatrix} \begin{pmatrix} x^*(t_1) \\ p(t_1) = -2Mx^*(t_1) \end{pmatrix},$$

thus

$$p(t) = -2 \underbrace{\left(-\tfrac{1}{2}\right)\left(\Phi_{21}(t,t_1) - 2\Phi_{22}(t,t_1)M\right)\left(\Phi_{11}(t,t_1) - 2\Phi_{12}(t,t_1)M\right)^{-1}}_{P(t):=} x^*(t).$$

If $P(t)$ exists in $[t_0, t_1]$ then

$$u^*(t) = -R^{-1}(t)B^\top(t)P(t)x^*(t).$$

For the existence of P : *Ricatti differential equation* (see *e.g.* [Liberzon, 2011]).

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Application of the PMP: the shooting method

Let us consider the problem

$$
(P) \qquad \begin{aligned} \max \Psi(x(T)), & \\ & \dot{x}(t) = f(t, x(t), u(t)), \\ & x(0) = x_0, \\ & u(t) \in U. \end{aligned}
$$

Let us suppose that the local optimum (x^*, u^*) verifies the following property.

Reduction hypothesis: *from the maximum condition of the PMP*

$$u^*(t) \in \operatorname{argmax}_{\omega \in U} H(t, x^*(t), \omega, p^*(t)),$$

one can write u^* as a function of x and p, this is

$$u^*(t) = \Upsilon(t, x(t), p(t)),$$

for (x, p) sufficiently close from (x^*, u^*) and $\Upsilon : I\!R^n \times I\!R^{n*} \to I\!R^m$.

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Comments on the Reduction hypothesis

- In the example of the factory, we have:

$$u(t) = \begin{cases} 1 & p(t) > 1, \\ 0 & p(t) \le 1. \end{cases} = \Upsilon(p(t)).$$

Remark: Here Υ is not continuous, but this is not an inconvenient.

- Also for the linear-quadratic regulator this hypothesis is satisfied:

$$\min \int_0^T \big(x^\top(t)Q(t)x(t) + u^\top(t)R(t)u(t)\big)\mathrm{d}t + x^\top(T)Mx(T),$$
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The maximum condition of the PMP implies

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Reduced system

We eliminate u from the formulation of the PMP and obtain:

$$
\text{(TPBVP)} \qquad
\begin{aligned}
&\dot{x}(t) = f\big(t, x(t), \Upsilon(t, x(t), p(t))\big),\\
&\dot{p}(t) = -p(t) D_x f\big(t, x(t), \Upsilon(t, x(t), p(t))\big),\\
&x(0) = x_0, \qquad \text{[initial condition on } x\text{]},\\
&p(T) = \nabla\Psi(x(T)), \qquad \text{[final condition on } p\text{]}.
\end{aligned}
$$

Thanks to the PMP, one transforms the optimal control problem in a **two-point boundary value problem (TPBVP)**.

Shooting function

In order to solve (TPBVP) we have to find $p_0 \in \mathbb{R}^{n*}$ such that the solution (x, p) of

$$\text{(SYS)} \qquad \begin{aligned} &\dot{x}(t) = f\big(t, x(t), \Upsilon(t, x(t), p(t))\big), \\ &\dot{p}(t) = -p(t) D_x f\big(t, x(t), \Upsilon(t, x(t), p(t))\big), \\ &\big(x(0), p(0)\big) = (x_0, p_0), \end{aligned}$$

satisfies the final condition

$$p(T) = \nabla\Psi(x(T)).$$

We define the **shooting function**:

$$\mathcal{S} : \mathbb{R}^{n*} \to \mathbb{R}^{n*}, \quad p_0 \mapsto p(T) - \nabla\Psi(x(T)),$$

where (x, p) is the solution of (SYS) with initial conditions x_0 and p_0.

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Shooting algorithm

Solving (TPBVP) is equivalent to **solving the nonlinear equation in finite dimension:**

$$\mathcal{S}(p_0) = 0, \quad \mathcal{S} : \mathbb{R}^{n,*} \to \mathbb{R}^{n,*}.$$

Remark: *In some cases in which Υ is not regular, for instance discontinuous, the problem is solved, by considering the "switching times" (times of jump) as additional variables of the function $\mathcal{S}$.*

Algorithm

Use Newton's method to solve $\mathcal{S}(p_0) = 0$.

Given an initial estimate $p_0^0 \in \mathbb{R}^{n,*}$, the Newton iteration:

$$p_0^{k+1} = p_0^k - (D\mathcal{S}(p_0^k))^{-1}\mathcal{S}(p_0^k).$$

Bonnans, J.F.
The shooting approach to optimal control problems.
In IFAC Proceedings Volumes, 2013

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Vaccination and treatment problem by the shooting method

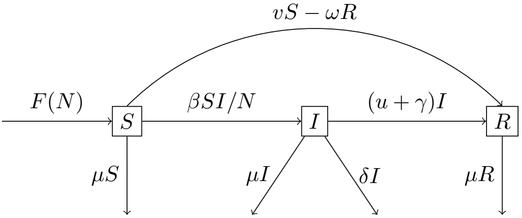


Figure: Model SIRS with vaccination v and treatment u

$$S + I + R = N.$$

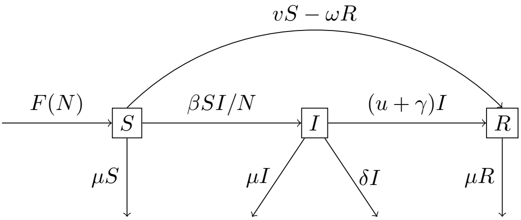
Dynamics:

$$\begin{aligned}
\dot{N} &= F(N) - \delta I - \mu N,\\
\dot{S} &= F(N) - \beta \frac{IS}{N} - vS + \omega R - \mu S,\\
\dot{I} &= \beta \frac{IS}{N} - (\gamma + \delta + u)I - \mu I,\\
\dot{R} &= vS + (\gamma + u)I - \omega R - \mu R.
\end{aligned}$$

Cost function

$$\int_0^T \left(B_1 I + B_2 v + B_3 u^2\right) \mathrm{d}t$$

Vaccination and treatment problem by the shooting method



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\dot{R} &= vS + (\gamma + u) I - \omega R - \mu R.
\end{aligned}$$

Cost function

$$\int_0^T \left(B_1 I + B_2 v + B_3 u^2 \right) \mathrm{d}t$$

Vaccination and treatment problem (continuation)

We get the optimal control problem:

$$\begin{aligned} \text{minimize} \quad & \int_0^T \left(B_1 I + B_2 v + B_3 u^2\right) \mathrm{d}t \\ \text{subject to} \quad & \dot{N} = F(N) - \delta I - \mu N, \\ & \dot{S} = F(N) - \beta \tfrac{IS}{N} - vS + \omega(N - S - I) - \mu S, \\ & \dot{I} = \beta \tfrac{IS}{N} - (\gamma + \delta + u) I - \mu I, \\ & 0 \leq v(t) \leq v_{\max}, \quad \text{a.e. on } [0, T] \\ & 0 \leq u(t), \quad \text{a.e. on } [0, T] \\ & N(0) = N_0, \quad S(0) = S_0, \quad I(0) = I_0, \quad C(0) = 0. \end{aligned}$$

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Vaccination and treatment problem: numerical solution

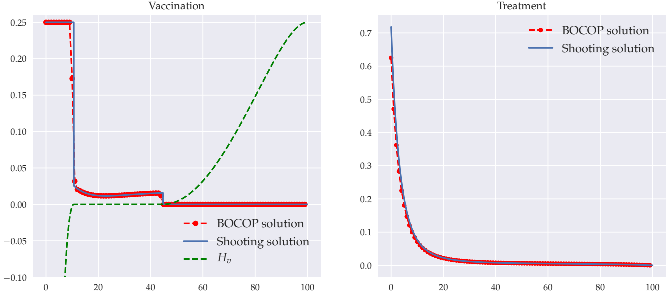

Figure: Optimal vaccination and treatment policies.

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The shooting algorithm for partially control-affine problems with application to an SIRS epidemiological model
In preparation, 2020

The problem with final constraints

$$
\text{(PF)} \qquad
\begin{aligned}
\max \ & \Psi(x(T)), \\
\text{s.t. } & \dot{x}(t) = f(t, x(t), u(t)), \quad \text{a.e. } t \in [0, T], \\
& x(0) = x_0, \\
& x(T) \in \mathcal{T} \subseteq I\!R^n, \quad \text{[final constraint]} \\
& u(t) \in U \subseteq I\!R^m.
\end{aligned}
$$

$\mathcal{T}$ is a closed set.

Theorem (Pontryagin Maximum Principle with final constraints)

Let u^* be an L^1-local minimum for (PF)*, x^* be the associated trajectory.*

Let C be a **Boltyanski approximating cone** *of the target set $\mathcal{T}$ at $x^*(T)$.*

Then, there exist an absolutely continuous function $p : [0, T] \to I\!R^{n,*}$, a scalar multiplier $\lambda \geq 0$ such that $(p, \lambda) \neq 0$, p is solution of the adjoint equation

$$\dot{p}(t) = -H_x(t, x^*(t), u^*(t), p(t)),$$

with the transversality condition

$$p(T) \in C^\perp + \lambda \nabla \Psi(x^*(T)),$$

and the maximum condition holds:

$$H(t, x^*(t), u^*(t), p(t)) = \max_{\omega \in U} H(t, x^*(t), \omega, p(t)), \quad \text{for a.e. } t \in [0, T].$$

Definitions

Polar cone

$$C^{\perp} := \{q \in \mathbb{R}^{n,*} : \langle q, y\rangle \leq 0, \quad \text{for all } y \in C\}.$$

Approximating cone

Given $\mathcal{T} \subset \mathbb{R}^n$ a set, $x \in \mathcal{T}$, we say a convex cone $C \subset \mathbb{R}^n$ is an *Boltyanski approximating cone* of $\mathcal{T}$ at x if there exists a neighbourhood of zero $W \subset \mathbb{R}^n$ and a continuous application $G\colon W \cap C \to \mathcal{T}$, such that

$$G(v) = x + v + o(|v|).$$

Route map of the proof

The *"profitable set"*:

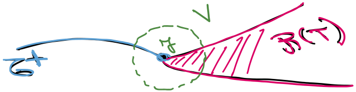
$$\mathcal{T}^+ := \{x \in \mathcal{T} : \Psi(x) > \Psi(x^*(T))\} \cup \{x^*(T)\}.$$

The *"reachable set"*:

$$\mathcal{R}(T) := \{x(T) : x \text{ corresponding to } u \colon [0,T] \to U \text{ with } x(0) = x_0\}.$$

The local optimality of (x^*, u^*) implies that

$\mathcal{T}^+$ and $\mathcal{R}(T)$ are locally separated[1] at $x^*(T)$.



CLAIM: This local separation implies the existence of the multiplier λ.

[1] *Given A_1, A_2 two subsets of a topological space, with $y \in A_1 \cap A_2$, A_1 and A_2 are* locally separable at y, *if there exists a neighbourhood V of y such that* $A_1 \cap A_2 \cap V = \{y\}$.

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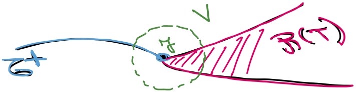

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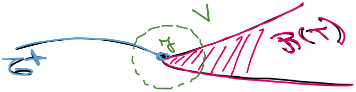

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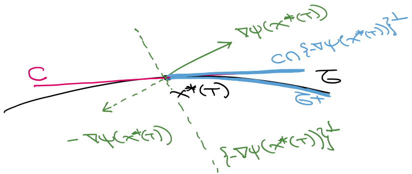

As an approximating cone of the reachable set we consider:

$$\Gamma := \mathrm{span}^+ \left\{ M(T,\tau)\Big[f(\tau, x^*(\tau), \omega) - f(\tau, x^*(\tau), u^*(\tau))\Big] : \omega \in U, 0 < \tau < T \right\},$$

where $M(\cdot,\cdot)$ is the fundamental matrix of the variational eq. $\dot{v} = D_x f v$.

Remarks:

- The key point of the proof is showing that Γ is an approximating cone of the reachable set $\mathcal{R}(T)$.
- It is necessary to combine several variations, and prove that they are "additive" at the final point.
- The local separation of $\mathcal{T}^+$ and $\mathcal{R}(T)$ implies the separation of their approximating cones $C \cap \{-\nabla\Psi(x^*(T))\}^\perp$ and Γ.

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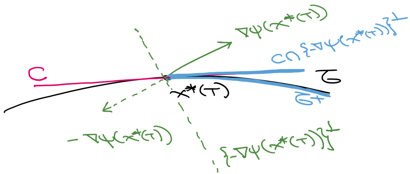

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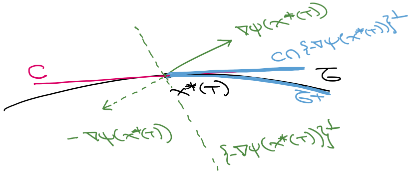

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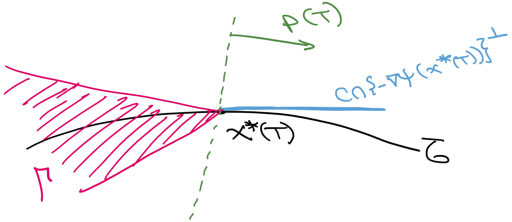
Finally, the Pontryagin Maximum Principle with final constraints follows from the separation of Γ and $C \cap \{-\nabla\Psi(x^*(T))\}^\perp$. More precisely,

$$p(T) \text{ is such that } \begin{cases} p(T) \cdot v \leq 0, & \forall v \in \Gamma, \\ p(T) \cdot v \geq 0, & \forall v \in C \cap \{-\nabla\Psi(x^*(T))\}^\perp. \end{cases}$$



■

Examples with final constraints

If there is enough time at the end, we will solve a problem with final constraints (and transversality condition) in the impulsive framework of next section.

For other nice worked-out examples see *e.g.*: resource extraction and sale problem in page 230 of [Leonard *et al.*, 1992], also the Moon Lander problem in page 55 of [Evans, 1983].

Leonard, D., Van Long, N. and Ngo, V.L.,
Optimal control theory and static optimization in economics.
Cambridge University Press, 1992

Evans, L.C.,
An introduction to mathematical optimal control theory - Version 0.2.
Lecture notes available at link

A quick remark for problems with state constraints

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& x(0) = x_0, \\
& g(t, x(t)) \le 0, \quad \text{for all } t \in [0,T] \quad \text{[state constraint]} \\
& u(t) \in U \subseteq \mathbb{R}^m.
\end{aligned}
$$

$t \mapsto g(t, x(t))$ is continuous. One takes the space of functions of bounded variation $BV(0,T)$ for the multiplier of the state constraint.

Loosely speaking, the PMP is modified in the following way: *there exist p and μ of bounded variation such that the adjoint equation*

$$
\mathrm{d}p(t) = -D_x H(t, x^*(t), u^*(t), p(t)) - D_x g(t, x^*(t))\mathrm{d}\mu(t),
$$

is satisfied, μ verifies the following complementarity relations:

$$
\mathrm{d}\mu \ge 0, \qquad \int_{[0,T]} g(t, x^*(t))\mathrm{d}\mu(t) = 0,
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and the maximum condition for the Hamiltonian holds.

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Some references for state-constrained problems

Bonnans, J.F.,
Course on Optimal Control - Part I: the Pontryagin approach.
Lecture notes available here, 2019

Vinter, R.,
Optimal control.
Springer Science & Business Media, 2010

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Geometric optimal control: theory, methods and examples.
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A strong version of the Lojasiewicz maximum principle.
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H. Sussmann,
Geometry and optimal control.
In Mathematical Control Theory, Springer, New York, 1999

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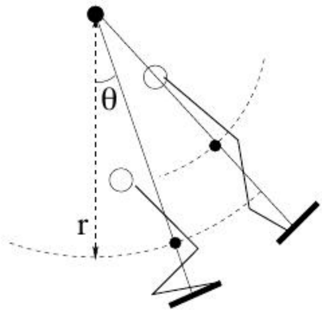
Impulsive systems, one motivating example: a swing

[Aldo and Alberto Bressan & Rampazzo, late 80s]

Boy riding swing: **changes (controls) the radius of oscillation**

$\theta(t)$: angle, $r(t)$: radius of oscillation



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Example of the swing: equations of motion

Setting $\omega := \dot{\theta}$ for the angular velocity, $u(t) := r(t)$ for the control, one can prove (see [Bressan & Piccoli, 2007]) that the equations of motion are given by

$$\dot{\theta} = \omega,$$
$$\dot{\omega} = -\frac{g \sin \theta}{u} - 2\frac{\omega}{u}\dot{u}.$$

If we consider the problem of maximizing the angle

$$\max |\theta(T)|$$

we find that the optimal control u^* jumps at the position $\theta = 0$!

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1. Introduction - Some simple examples
2. Simplified version of Pontryagin's Maximum Principle: no terminal constraints
   - Statement and proof
   - Examples
   - The linear-quadratic regulator
   - Application: shooting method
3. Pontryagin's Maximum Principle with final constraints
   - Statement and proof
   - Further considerations
4. Impulsive Optimal Control problems
   - Introductory example
   - **Impulsive control systems**
   - A Higher-order Pontryagin Maximum Principle for impulsive problems: statement and sketch of the proof
   - Application of the Higher-order PMP: time for an example?

Different formulation for optimal control problems

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\begin{aligned}
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&\qquad \dot{u}(t) \in \mathcal{C}, \quad a(t) \in A,
\end{aligned}
$$

where $u : [0, T] \to I\!R^m$, $a : [0, T] \to I\!R^l$ are the control variables.

$\mathcal{C} \subseteq I\!R^m$ is a **convex cone**, $A \subset I\!R^l$ is usually a **compact** subset.

Standard case (regular control, as we have seen up to here): for $u \in AC$, $a \in L^1$, the trajectory x is the solution of the ODE in the classical sense of Carathéodory.

QUESTION: how do we represent the situation of the example?

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Considering less regular u...

Motivations to study discontinuous trajectories:

1) The optimization may force an instantaneous change in the trajectory x, for instance, if there is lack of coercivity of the cost,

2) large change of the value of the trajectory, in a short amount of time.

Some examples in:

Azimov, D. and Bishop R.,
New trends in astrodynamics and applications: Optimal trajectories for space guidance.
Annals NY Acad. Sciences, 2005

Catllá et al.,
On Spiking models for synaptic activity and impulsive differential equations
SIAM Review, 2005

Gajardo, P., Ramirez, H. and Rapaport, A.,
Minimal time sequential batch reactors with bounded and impulse controls.
SIAM J. Control Opt., 2007

The *impulsive control system*: discontinuous u, or worse

Let us consider now the control system

$$\dot{x}(t) = f(x(t), a(t)) + \sum_{i=1}^{m} g_i(x(t))\dot{u}_i(t),$$

with $u \in BV([0,T]; I\!R^m)$, $a \in L^1([0,T]; A)$, then...

...what do we mean with x **"solution" of the above equation**?

How do we represent impulsive solutions?

When $u \in AC$, we can define

$$s \colon [0,T] \to [0,1], \quad s(t) := \frac{t + \mathrm{Var}_{[0,t]}(u)}{T + \mathrm{Var}_{[0,T]}(u)}.$$

Then s is continuous and strictly increasing, and it has a Lipschitz-continuous strictly increasing inverse φ_0.

Reparametrization of time: $\varphi_0 : [0,1] \to [0,T]$, $\varphi_0 := s^{-1}$,

$$\varphi(s) := u \circ \varphi_0(s), \quad \alpha(s) := a \circ \varphi_0(s), \quad y(s) := x \circ \varphi_0(s).$$

Then (φ_0, φ, y) is solution of the space-time system

$$\begin{cases} {y_0}'(s) = \underbrace{{\varphi_0}'(s)}_{>0}, \\ y'(s) = f(y(s), \alpha(s)) \underbrace{{\varphi_0}'(s)}_{>0} + \displaystyle\sum_{i=1}^{m} g_i(y(s)) {\varphi_i}'(s), & s \in [0,1]. \\ (y_0, y)(0) = (0, x_0), \end{cases}$$

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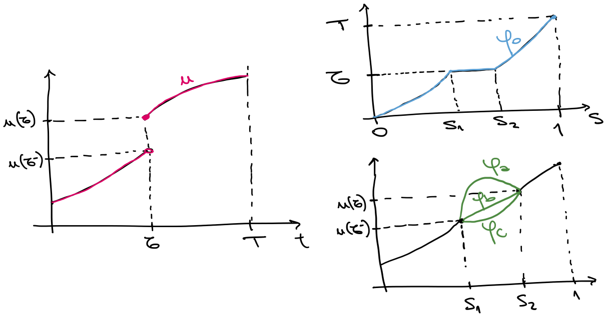
Let us extend the idea for $u \in BV$: *graph completion*

Given $u \in BV$, (φ_0, φ) is a **graph completion** of u if:

- $(\varphi_0, \varphi) : [0, S] \to [0, T] \times \mathbb{R}^m$ is Lipschitz-continous, φ_0 non-decreasing,
- $\forall \tau \in [0, T]$, there exists $s \in [0, S]$ such that $(\tau, u(\tau)) = (\varphi_0, \varphi)(s)$.



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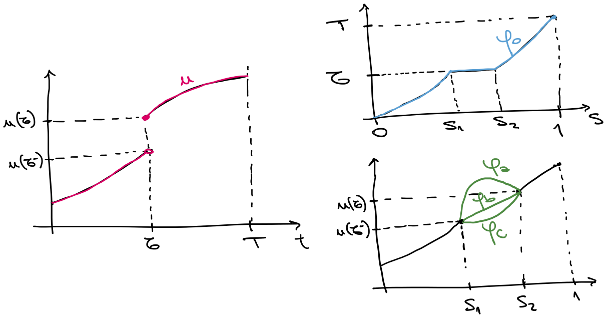

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Extended system

Given $u \in BV$ and a Lipschitz continuous graph completion $(\varphi_0, \varphi) : [0, S] \to [0, T] \times I\!\!R^m$, let y be the solution of the extended system:

$$\begin{cases} {y_0}'(s) = \underbrace{{\varphi_0}'(s)}_{\geq 0}, \\ y'(s) = f(y(s), \alpha(s)) \underbrace{{\varphi_0}'(s)}_{\geq 0} + \sum_{i=1}^{m} g_i(y(s)) {\varphi_i}'(s), \\ (y_0, y)(0) = (0, x_0)\,, \end{cases} \quad s \in [0, S].$$

Note that $(\varphi_0', \varphi') \in L^\infty([0, S]; I\!\!R^{m+1})$.

Remark

The intervals $[s_1, s_2] \subset [0, S]$ with $\varphi_0' = 0$ are the intervals of impulse.

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Note that $(\varphi_0', \varphi') \in L^\infty([0, S]; I\!R^{m+1})$.

Remark

The intervals $[s_1, s_2] \subset [0, S]$ *with* $\varphi_0' = 0$ *are the intervals of impulse.*

The *impulsive* optimal control problem

Set $(w_0, w) := (\varphi_0', \varphi')$. We get the problem:

$$
(P_e) \qquad
\begin{aligned}
\max \ & \Psi\big(y_0(S), y(S)\big) \\
\text{s.a.} \ & {y_0}'(s) = w_0(s), \\
& y'(s) = f(y(s), \alpha(s)) w_0(s) + \sum_{i=1}^{m} g_i(y(s)) w_i(s), \\
& (y_0, y)(0) = (0, x_0), \quad \big(y_0, y\big)(S) \in \mathcal{T}, \\
& w_0(s) \geq 0, \quad w(s) \in \mathcal{C}, \quad \alpha(s) \in A,
\end{aligned}
$$

where $(w_0, w) \in L^\infty([0, S]; I\!R^{1+m})$, $\alpha \in L^1([0, S]; I\!R^l)$.

Remark: When $w_0 = 0$, only the **impulsive part of the system** acts:

$$
\dot{y}(s) = \sum_{i=1}^{m} g_i(y(s)) w_i(s).
$$

Impulsive perturbations through the use of Lie brackets

Notation: for a vector field $g \in C^\infty(I\!R^m)$, we write $\exp(tg)(x)$ to denote $X(t)$, where $\dot{X} = g(X),\ X(0) = x$.

Asymptotic formula:

$$\exp(-tg_2) \circ \exp(-tg_1) \circ \exp(tg_2) \circ \exp(tg_1)(x) = x + t^2 [g_1, g_2](x) + o(t^2),$$

where

$[g_1, g_2](x) := Dg_2(x)g_1(x) - Dg_1(x)g_2(x)$ is the Lie bracket of g_1, g_2

For simplicity of the presentation, we assume for simplicity that $g_1, \ldots, g_m \in C^\infty(I\!R^n)$ and that the cone $\mathcal{C}$ is $I\!R^m$.

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Lemma (Construction of impulsive variations)

Given a compact subset $K \subseteq \mathbb{R}^n$ and a natural number M, there exist $c, \bar{s} > 0$ such that: for all $x \in K$, Lie bracket B of order h, with $h \leq M$, all time $0 < s < \bar{s}$, we can construct a control

$$w_{B,s} \colon [0, s] \to \{\pm \mathbf{e}_1, \ldots, \pm \mathbf{e}_m\}$$

such that

(i) $w_{B,s}$ *is piecewise continuous,*

(ii) *it holds*

$$\left| y[x, w_{B,s}](s) - x - \frac{B(x)}{r^h} s^h \right| \leq \frac{c}{r^h} s^{h+1},$$

where $y[x, w_{B,s}]$ is the trajectory starting at x and associated to the control $w_{B,s}$, and $r = r(B)$ is increasing with the order h.

Shortly speaking:

$$y[x, w_{B,s}](s) - x \ \text{ has the direction of } B(x) + o(s^h).$$

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Theorem (Higher-order Maximum Principle for an impulsive problem)

Let $(\bar{S}, \bar{y}_0, \bar{y}, \bar{w}_0, \bar{w}, \bar{\alpha})$ *be a local minimizer for the extended problem* (P_{e})*, and let* C *be an approximating cone of* $\mathcal{T}$ *at* $(\bar{y}_0, \bar{y})(\bar{S})$.

Then the following conditions hold: the exist $p \in AC\left([0, \bar{S}]; I\!R^n\right)$*, real numbers* p_0 *and* $\lambda \geq 0$ *such that* $(p_0, p, \lambda) \neq (0, 0, 0)$*, and*

- **Adjoint equation:**

$$\frac{dp}{ds} = -p \cdot \left(\frac{\partial f}{\partial x}(\bar{y}, \bar{\alpha})\bar{w}_0 + \sum_{i=1}^{m} \frac{\partial g_i}{\partial x}(\bar{y})\bar{w}_i \right),$$

- **Transversality condition:**

$$(-p_0, -p(\bar{S})) \in C^{\perp} + \lambda \nabla \Psi((\bar{y}_0, \bar{y})(\bar{S})).$$

it continues. . .

Theorem (...continuation)

- **Maximum condition:** *for a.a.* $s \in [0, S]$,

$$p(s) \cdot \left(f(\bar{y}(s), \bar{\alpha}(s))\bar{w}_0(s) + \sum_{i=1}^{m} g_i(\bar{y}(s))\bar{w}_i(s) \right) + p_0\, \bar{w}_0(s) =$$

$$\max_{\substack{(\omega_0,\omega)\in \mathbb{R}_+\times\mathcal{C} \\ a\in A}} \left\{ p(s) \cdot \left(f(\bar{y}(s), a)\omega_0 + \sum_{i=1}^{m} g_i(\bar{y}(s))\omega_i \right) + p_0\, \omega_0 \right\} = 0.$$

- **Higher-order conditions:**

For any Lie bracket B *of* $g_1, \ldots, g_m$,

$$p(s) \cdot B(\bar{y}(s)) = 0, \qquad \textit{for a.a. } s \in [0, \bar{S}].$$

End of statement.

Remarks

- The novelty are the constructions of the impulsive variations, and their composition with needle variations.
- We found examples in which the (first order) Pontryagin's Maximum Principle and well-established second-order conditions do not rule out a non-optimal solution, but our Higher-order result does.

Example

$$
\begin{aligned}
\min \;& x_3(1) + x_4(1), \\
& \dot{x} = f(x) + g_1(x)\dot{u}_1 + g_2(x)\dot{u}_2, \\
& x(0) = (1,0,0,0,0), \\
& (x_1, x_2)(1) = (0,0), \; x_3(1) \in [-1, +\infty),
\end{aligned}
$$

$$
f = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2}\left(x_2^2 + x_3^2 + (1 - x_1 - x_5)^2\right) \\ 1 \end{pmatrix}, \; g_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2}(x^2)^2 \\ 0 \\ 0 \end{pmatrix}, \; g_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
$$

- $x_4 \geq 0$, and the optimal solution should have $x_3(1) = -1$, $x_4(1) = 0$.
  To reach $x_3 = -1$ we have to use x_2, and then $x_4(1) > 0$!
- Then there is no regular solution with $x_3(1) = -1$, $x_4(1) = 0$.
- We will reach $x_3(1) = -1$ **instantaneously!**

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Example (continuation)

The reparametrized problem on $[0, S]$ is

$$\begin{aligned} \min\ & y_3(S) + y_4(S), \\ & y_0' = w_0, \\ & y' = f(y)w_0 + g_1(y)w_1 + g_2(y)w_2, \\ & (y_0, y)(0) = (0, 1, 0, 0, 0, 0), \\ & (y_0, y)(S) \in \{(1, 0, 0)\} \times [-1, +\infty) \times I\!R^2 =: \mathcal{T}. \end{aligned}$$

Example (continuation)

Let us consider two controls: an optimal one (w_0^*, w^*) that remains with $x_2^* = x_3^* = 0$ until $t = 1$, when it jumps to $(0, 0, -1, 0, 1)$; and a non-optimal one $(\hat{w}_0, \hat{w})$.

And let us see that (w_0^*, w^*) satisfies the Higher-order PMP, and $(\hat{w}_0, \hat{w})$ satisfies the first-order PMP but ir does not verify the higher-order Lie bracket conditions.

Take $[0, S] = [0, 2]$. And define

$$w_0^*(s) := \begin{cases} 1, \text{ on } [0,1], \\ 0, \text{ on } [1,2], \end{cases} \quad \begin{pmatrix} w_1^* \\ w_2^* \end{pmatrix}(s) := \begin{cases} -\mathbf{e}_1, & \text{on } [0,1], \\ -4\sqrt[3]{2}\,\mathbf{e}_1, & \text{on } (1, 5/4], \\ -4\sqrt[3]{2}\,\mathbf{e}_2, & \text{on } (5/4, 3/2], \\ 4\sqrt[3]{2}\,\mathbf{e}_1, & \text{on } (3/2, 7/4], \\ 4\sqrt[3]{2}\,\mathbf{e}_2, & \text{on } (7/4, 2]. \end{cases}$$

Example (continuation)

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Example (continuation)

We have

$$y_4^* = 0, \quad \text{on } [0,2],$$

$$y_3^*(s) = \begin{cases} 0, & \text{on } [0,3/2], \\ -4(s-3/2), & \text{on } (3/2,7/4], \\ -1, & \text{on } (7/4,2] \implies y_3^*(2) = -1. \end{cases}$$

We can see that with

$$\lambda := 1, \quad p_0 := 0, \quad p := (0,0,0,-1,0)$$

the Higher-order PMP is satisfied.

In fact, the fourth component of every element B of the Lie algebra generated by $\{g_1, g_2\}$ is always 0.

Example (continuation)

Let us define the second control $(\hat{w}_0, \hat{w})$:

$$\begin{pmatrix} \hat{w}_0 \\ \hat{w}_1 \\ \hat{w}_2 \end{pmatrix}(s) := \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}, \quad \text{on } [0,2].$$

We have $\hat{y}_3(2) = 0!$

The transversality and the maximum conditions yield:

$$\hat{p} = (0, 0, \hat{\lambda}, \hat{\lambda}, 0),$$

with $\hat{\lambda} > 0$.

With $\hat{\lambda} = 1$ the maximum condition of the first-order PMP is verified.

But, since

$$\big[[g_1, g_2], g_2\big] = (0, 0, -1, 0, 0)^\top,$$

"$pB = 0$" implies $\hat{p}_3 \equiv 0$, $\hat{\lambda} = 0$, and then $(\hat{p}_0, \hat{p}, \hat{\alpha}) = 0$.

There is **no nontrivial multiplier verifying the Higher-order PMP.**

□

Outline

References for this section

M.S. Aronna, M. Motta and F. Rampazzo,
Necessary higher-order optimality conditions involving Lie brackets for impulsive optimal control problems.
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