The Strong Homotopy Structure of Poisson Reduction

joint work with Chiara Esposito and Andreas Kraft

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Aim of the Talk

Outline

1. Marsden-Weinstein reduction and the BRST-method
2. L_∞-algebras and curved Lie algebras
3. Equivariant Multivector Fields and their Maurer-Cartan Elements
4. Construction of the L_∞-morphism

Marsden-Weinstein Reduction

Definition

A Hamiltonian G-space is a quintuplet (M, Φ, G, ω, J) with

- a manifold M, a Lie group G and a (left) action $\Phi\colon G \times M \to M$
- an invariant symplectic structure ω
- a linear equivariant map $J\colon \mathfrak{g} \to \mathscr{C}^\infty(M)$ (momentum map) ($\Leftrightarrow$ $J\colon M \to \mathfrak{g}^*$ equivariant)

such that

$$\xi_M := \frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\exp(t\xi)} = -X_{J(\xi)} \quad \text{and} \quad J([\xi,\eta]) = \{J(\xi), J(\eta)\}_\omega$$

Example

Lie group action $\Phi\colon G \times M \to M \rightsquigarrow$ cotangent lift $T^*\Phi\colon G \times T^*M \to T^*M$ +

- ω_{can}
- $J\colon \mathfrak{g} \ni \xi \mapsto (\alpha_p \mapsto \alpha_p(\xi_M(p))) \in \mathscr{C}^\infty(M)$

Marsden-Weinstein Reduction

Theorem (Marsden, Weinstein)

(M, Φ, G, ω, J) Hamiltonian G-space + $\iota\colon C := J^{-1}(0) \hookrightarrow M$ submanifold + $p\colon C \to M_{\mathrm{red}} := C/G$ manifold. Then $\exists!$ symplectic structure $\omega_{\mathrm{red}} \in \Omega^2(M_{\mathrm{red}})$, such that

$$\iota^*\omega = p^*\omega_{\mathrm{red}}.$$

Remark

If we relax symplectic to Poisson the theorem still holds replacing pull-backs by backward Dirac maps.

Formal Hamiltonian Actions

(M, Φ, G, π, J) Hamiltonian G-space

$$(\pi, J) \rightsquigarrow (\pi_\hbar, J_\hbar)$$

That is $\pi_\hbar \in \Gamma^\infty(\Lambda^2 TM)^G[[\hbar]]$ formal Poisson (i.e. $[\![\pi_\hbar, \pi_\hbar]\!] = 0$), formal momentum map $J_\hbar \colon \mathfrak{g} \to \mathscr{C}^\infty(M)[[\hbar]]$ with

$$\xi_M = -\pi_\hbar^\sharp(\mathrm{d}J_\hbar(\xi)) \quad \text{(The group action is not changed)}$$

Remark

A pair $(\pi_\hbar, J_\hbar)$ can be seen as a deformation of (π_0, J_0) which is a Poisson structure with momentum map.

Problem

Marsden-Weinstein reduction in this form does not really apply for formal Poisson and formal momentum maps.

Reminder: BRST-method

$(M, \Phi, G, \pi_\hbar, J_\hbar)$ formal Hamiltonian G-space.BRST-algebra with the component-wise graded Product:

$$\mathscr{A}^{(\bullet)} := \Lambda^\bullet \mathfrak{g}^* \otimes \Lambda^{-\bullet} \mathfrak{g} \otimes \mathscr{C}^\infty(M)[[\hbar]]$$

+

- degree 0 Poisson structure $\{\cdot\,,\,\cdot\}$ with

$$\{\alpha \otimes f, \beta \otimes g\}_\hbar = \{\alpha, \beta\}_\mathfrak{g} \otimes fg + \alpha\beta \otimes \{f, g\}_{\pi_\hbar}$$

- degree +1 charge $\Theta = -\frac{1}{2}[\,\cdot\,,\,\cdot\,] + J_\hbar$ with $\{\Theta, \Theta\} = 0 \implies \mathrm{d}_\Theta = \{\Theta, \cdot\,\}$ is a differential

Theorem

$(\mathrm{H}^0_{\mathrm{d}_\Theta}(\mathscr{A}^{(\bullet)}), \{\,\cdot\,,\,\cdot\,\}) \cong (\mathscr{C}^\infty(M_{\mathrm{red}})[[\hbar]], \{\,\cdot\,,\,\cdot\,\}_{\mathrm{red},\hbar})$ (in nice cases). Where $\{\,\cdot\,,\,\cdot\,\}_{\mathrm{red},0}$ is the M-W-reduction of $\{\,\cdot\,,\,\cdot\,\}_{\pi_0}$ with respect to J_0.

"Summary"

For fixed (M, Φ, G, J), we have a map

$$\left\{ \begin{array}{c} (\pi_\hbar, J_\hbar), \text{ formal Poisson + formal} \\ \text{momentum map on } M\text{, s.t. } J_\hbar = J + \mathcal{O}(\hbar) \end{array} \right\} \longrightarrow \left\{ \pi_{\mathrm{red},\hbar} \text{ formal Poisson on } M_{\mathrm{red}} \right\}$$

Question

What (algebraic) structure does this map have?

The sets on both sides do not posses any (obvious, linear) algebraic structures, but:

Theorem

Both sets are can be identified with Maurer-Cartan elements of certain curved Lie algebras and there is an L_∞-morphism between them inducing the above map on the level of Maurer-Cartan elements.

Curved Lie algebras and L_∞-algebras

Definition

A curved Lie algebra is a graded vector space $\mathfrak{L}^\bullet = \bigoplus_{i\in\mathbb{Z}} \mathfrak{L}^i$ together with

1. a graded Lie bracket $[\,\cdot\,,\,\cdot\,]\colon \mathfrak{L}^\bullet \times \mathfrak{L}^\bullet \to \mathfrak{L}^\bullet$ of degree 0
2. a derivation $\mathrm{d}\colon \mathfrak{L}^\bullet \to \mathfrak{L}^\bullet$ of $[-,-]$ of degree 1
3. an element (the curvature) $R \in \mathfrak{L}^2$

such that

1. $\mathrm{d}R = 0$
2. $\mathrm{d}^2 = [R,\,\cdot\,]$

If $R = 0$, we say that $(\mathfrak{L}, \mathrm{d}, [\,\cdot\,,\,\cdot\,])$ is a differential graded Lie algebra (=DGLA).

Definition

A degree $+1$ coderivation Q on the co-unital conilpotent cocommutative coalgebra $\mathrm{S}(\mathfrak{L}[1]^\bullet)$ cofreely cogenerated by the graded vector space $\mathfrak{L}^\bullet[1]$ is called an L_∞-structure on the graded vector space $\mathfrak{L}^\bullet$ if $Q^2 = 0$. If $Q(1) = 0$ we say that $(\mathfrak{L}, Q)$ is *flat*.

Lemma

A L_∞-structure Q on $\mathfrak{L}^\bullet$ is completely determined by its *Taylor coefficients*

$$Q_n \colon \mathrm{S}^n(\mathfrak{L}[1]^\bullet) \to \mathfrak{L}^\bullet[2].$$

If $(\mathfrak{L}, Q)$ is flat then Q_1 is a differential.

Definition

Let $(\mathfrak{L}^\bullet, Q)$ be an L_∞-algebra. An element $\pi \in \mathfrak{L}^1 = \mathfrak{L}[1]^0$ is called Maurer–Cartan element, if

$$\sum_{k \geq 0} \frac{1}{k!} Q_k(\pi^{\vee k}) = 0$$

Example

A curved Lie algebra $(\mathfrak{L}^\bullet, R, d, [-,-])$ induces an L_∞-structure Q on $\mathfrak{L}^\bullet$ by $Q_0(1) = -R$, $Q_1 = -\mathrm{d}$ and $Q_2(\gamma \vee \mu) = -(-1)^{|\gamma|}[\gamma, \mu]$ for all $\gamma, \mu \in \mathfrak{L}[1]^\bullet$.

Curved Lie algebras and L_∞-algebras

Definition

Let $(\mathfrak{L}^\bullet, Q)$ and $(\mathfrak{K}^\bullet, Q')$ by L_∞-algebras. An L_∞-morphism is a degree 0 coalgebra morphism

$$F\colon \mathrm{S}(\mathfrak{L}[1]^\bullet) \to \mathrm{S}(\mathfrak{K}[1]^\bullet),$$

such that $F \circ Q = Q' \circ F$.

Lemma

An L_∞-morphism $F\colon (\mathfrak{L}^\bullet, Q) \to (\mathfrak{K}^\bullet, Q')$ is completely determined by its Taylor coefficients

$$F_n\colon \mathrm{S}^n(\mathfrak{L}[1]^\bullet) \to \mathfrak{K}[1].$$

and for a MC element $\pi \in \mathfrak{L}^1$, the element

$$\sum_{k\geq 1} \frac{1}{k!} F_k(\pi^{\vee k})$$

is a MC element.

Twisting of L_∞-algebras and their morphisms

For any $\pi \in \mathfrak{L}^1$ and any L_∞-morphism $F\colon (\mathfrak{L}^\bullet, Q) \to (\mathfrak{K}^\bullet, \widetilde{Q})$, such that

$$\widetilde{\pi} := \sum_{k\geq 1} \frac{1}{k!} F_k(\pi^{\vee k})$$

exists. If

- $Q_k^\pi = \sum_i \frac{1}{i!} Q_{i+k}(\pi^{\vee i} \vee \cdot\,)$
- $\widetilde{Q}_k^{\widetilde{\pi}} = \sum_i \frac{1}{i!} \widetilde{Q}_{i+k}(\widetilde{\pi}^{\vee i} \vee \cdot\,)$
- $F_k^\pi = \sum_i \frac{1}{i!} F_{i+k}(\pi^{\vee i} \vee \cdot\,)$

are also well-defined, then $(\mathfrak{L}^\bullet, Q^\pi)$ and $(\mathfrak{K}^\bullet, \widetilde{Q}^{\widetilde{\pi}})$ are L_∞-algebras and

$$F^\pi\colon (\mathfrak{L}^\bullet, Q^\pi) \to (\mathfrak{K}^\bullet, \widetilde{Q}^{\widetilde{\pi}})$$

is a L_∞-morphism.For curved Lie algebras the new structures are

1. $R^\pi = R + \mathrm{d}\pi + \frac{1}{2}[\pi, \pi]$
2. $\mathrm{d}^\pi = \mathrm{d} + [\pi, \cdot\,]$

L_∞-Quasi-Isomorphisms

Definition

Let $F\colon (\mathfrak{L}^\bullet, Q) \to (\mathfrak{K}^\bullet, \widetilde{Q})$ be a L_∞-morphism between two flat L_∞-algebras. We say that F is a L_∞-quasi-isomorphism, if F_1 is an isomorphism in cohomology.

Theorem

Let $F\colon (\mathfrak{L}^\bullet, Q) \to (\mathfrak{K}^\bullet, \widetilde{Q})$ be a L_∞-quasi-isomorphism between two flat L_∞-algebras. Then there exists an L_∞-quasi-ismorphism $G\colon (\mathfrak{K}^\bullet, \widetilde{Q}) \to (\mathfrak{L}^\bullet, Q)$, such that G_1 is a quasi-inverse of F_1 on the level of complexes.

Equivariant Multivector Fields

Definition

The graded vector space $T^\bullet_{\mathfrak{g}}(M)$ given by

$$T^k_{\mathfrak{g}}(M) = \bigoplus_{2i+j=k} (\mathrm{S}^i \mathfrak{g}^* \otimes T^j_{\mathrm{poly}}(M))^G$$

together with

- the bracket $[\,\cdot\,,\,\cdot\,]_{\mathfrak{g}} \colon T^k_{\mathfrak{g}}(M) \times T^\ell_{\mathfrak{g}}(M) \to T^{k+\ell}_{\mathfrak{g}}(M)$, given by

$$[P \otimes X, Q \otimes Y]_{\mathfrak{g}} := P \vee Q \otimes [X, Y]$$

- the curvature $\lambda = e^i \otimes (e_i)_M \in T^2_{\mathfrak{g}}(M)$

is a curved Lie algebra.

$$\begin{aligned} MC(T^\bullet_{\mathfrak{g}}(M), \lambda, [\,\cdot\,,\,\cdot\,]_{\mathfrak{g}}) :&= \{\Pi \in T^1_{\mathfrak{g}}(M) \mid \lambda + \frac{1}{2}[\Pi, \Pi] = 0\} \\ &= \{\Pi = \pi - J = \text{Poisson - momentum map}\} \end{aligned}$$

Equivariant Multivector Fields: What exactly do we want?

We choose a an equivariant momentum map $J\colon \mathfrak{g} \to \mathscr{C}^\infty(M)$ and consider

$$(T^\bullet_\mathfrak{g}(M)[[\hbar]], \hbar\lambda, \mathrm{d}^{-J}, [\,\cdot\,,\,\cdot\,]_\mathfrak{g}),$$

Maurer-Cartan elements: $\hbar(\pi_\hbar - J_\hbar) \in T^1_{\mathrm{poly}}(C)^G \oplus (\mathfrak{g}^* \otimes \mathscr{C}^\infty(M))^G$:
$\pi_\hbar$ is a formal Poisson structure + formal momentum map $J + \hbar J_\hbar$.

Main Aim

Find L_∞-morphism

$$MW_{\mathrm{red}}\colon (T^\bullet_\mathfrak{g}(M), \lambda, \mathrm{d}^{-J}, [\,\cdot\,,\,\cdot\,]_\mathfrak{g}) \to (T_{\mathrm{poly}}(M_{\mathrm{red}}), [\,\cdot\,,\,\cdot\,])$$

+ extend it $\hbar$-linearly to

$$MW_{\mathrm{red}}\colon (T^\bullet_\mathfrak{g}(M)[[\hbar]], \hbar\lambda, \mathrm{d}^{-J}, [\,\cdot\,,\,\cdot\,]_\mathfrak{g}) \to (T_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]], [\,\cdot\,,\,\cdot\,]).$$

Equivariant Multivecor Fields on $C \times \mathfrak{g}^*$

Assume $M = C \times \mathfrak{g}^*$ with

1. $\Phi\colon G \times M \ni (g, (c, \alpha)) \mapsto (\Phi^C_g(c), \mathrm{Ad}^*_{g^{-1}} \alpha) \in M$
2. $J\colon M \ni (c, \alpha) \mapsto \alpha \in \mathfrak{g}^*$
3. $\pi_{\mathrm{KKS}} \in T^1_{\mathrm{poly}}(M)$

This a Hamiltonian G-space with $J^{-1}(\{0\}) = C$.
By twisting with $\Pi := \pi_{\mathrm{KKS}} - J$ we get the DGLA:

$$(T^\bullet_{\mathfrak{g}}(M), \mathrm{d}^\Pi, [\,\cdot\,,\,\cdot\,])$$

(Intermediate) Aim

We want to find I and II and DGLA-morphisms with

$$(T^\bullet_{\mathfrak{g}}(M), \mathrm{d}^\Pi, [\,\cdot\,,\,\cdot\,]) \longrightarrow I \xleftarrow{\simeq} II \xrightarrow{\simeq} (T^\bullet_{\mathrm{poly}}(M_{\mathrm{red}}), [\,\cdot\,,\,\cdot\,])$$

I: Taylor expansion around C

The *vertical Taylor expansion*

$$\mathcal{T}\colon \mathscr{C}^\infty(C \times \mathfrak{g}^*) \ni f \mapsto \sum_{I \in \mathbb{N}_0^{\dim \mathfrak{g}}} \frac{1}{I!} e_I \otimes \iota^*\Big(\frac{\partial^{|I|} f}{\partial \alpha_I}\Big) \in \prod_{i \in \mathbb{N}_0} (S^i \mathfrak{g} \otimes \mathscr{C}^\infty(C))$$

is

- G-equivariant
- extendable to a Lie algebra morphism $\mathcal{T}\colon T^k_{\mathfrak{g}} \to T^k_{\mathfrak{g},\mathrm{Tay}} := \bigoplus_{2i+j+\ell} \big(\mathrm{S}^i \mathfrak{g}^* \otimes \prod_{n \in \mathbb{N}_0} (S^n \mathfrak{g} \otimes \Lambda^j \mathfrak{g}^* \otimes T^\ell_{\mathrm{poly}}(C))\big)^G$

Lemma

The map $\mathcal{T}$ is a

- DGLA morphism between $(T^\bullet_{\mathfrak{g}}(M), \mathrm{d}^\Pi, [\,\cdot\,,\,\cdot\,])$ and $(T^\bullet_{\mathfrak{g},\mathrm{Tay}}, \mathrm{d}^{\mathcal{T}(\Pi)}, [\,\cdot\,,\,\cdot\,])$
- a morphism of curved Lie algebras between $(T^\bullet_{\mathfrak{g}}(M), \lambda, \mathrm{d}^{-J}, [\,\cdot\,,\,\cdot\,])$ and $(T^\bullet_{\mathfrak{g},\mathrm{Tay}}, \mathcal{T}(\lambda), \mathrm{d}^{-\mathcal{T}(J)}[\,\cdot\,,\,\cdot\,])$

I: Taylor expansion around C

Interesting sub-DGLA:

$$(T^\bullet_{\mathrm{Cart}}(C) = \prod_{i\in\mathbb{N}_0} (\mathrm{S}^i\mathfrak{g}\otimes T^\bullet_{\mathrm{poly}}(C))^G, \partial, [\,\cdot\,,\,\cdot\,])$$

with simple differential

$$\partial\colon T^\bullet_{\mathrm{Cart}}(C) \ni P\otimes X \mapsto e^i(P)\otimes(e_i)_C\wedge X\in T^{\bullet+1}_{\mathrm{Cart}}(C)$$

Theorem

There is a homotopy $h\colon T^\bullet_{\mathfrak{g},\mathrm{Tay}}\to T^{\bullet-1}_{\mathfrak{g},\mathrm{Tay}}$, such that

$$(T_{\mathrm{Cart}}(C),\partial) \underset{p}{\overset{i}{\rightleftarrows}} (T_{\mathrm{Tay}}(C\times\mathfrak{g}^*), \mathrm{d}^{\mathcal{J}(\Pi)}) \circlearrowleft h \tag{1}$$

is a deformation retract. In particular, ι is a quasi-isomorphism.

Definition (Cartan model)

For a Lie group action $\Phi\colon G \times C \to C$, we call

$$(T^\bullet_{\mathrm{Cart}}(C) = \prod_{i \in \mathbb{N}_0} (\mathrm{S}^i \mathfrak{g} \otimes T^\bullet_{\mathrm{poly}}(C))^G, \partial, [\,\cdot\,,\,\cdot\,])$$

the *Cartan model*.

Assume C is a principal G-bundle $\implies$ C/G is a manifold and DGLA-map

$$p\colon T^\bullet_{\mathrm{Cart}}(C) \to T^\bullet_{\mathrm{poly}}(C)^G \to T^\bullet_{\mathrm{poly}}(C/G)$$

Aim

Want to show that p is a quasi-isomorphism. In fact, we construct a (family of) deformation retract(s) around p.

II: Cartan model

Choose principal connection $\omega \in (\Omega(C) \otimes \mathfrak{g})^G$ and define

$$\tilde{h}_\omega \colon T^\bullet_{\mathrm{Cart}}(C) \ni P \otimes X \mapsto e_i \vee P \otimes \mathrm{i_a}(\omega^i)X \in T^{\bullet-1}_{\mathrm{Cart}}(C).$$

Lemma

$$\partial \tilde{h}_\omega + \tilde{h}_\omega \partial = \deg_{\mathfrak{g}} + \deg_{\mathrm{vert}}$$

+ rescaled:

$$T_{\mathrm{poly}}(C/G) \underset{p}{\overset{\mathrm{hor}_\omega}{\rightleftarrows}} T_{\mathrm{Cart}}(C) \circlearrowleft h_\omega$$

is a deformation retract for all principal connections $\omega \in (\Omega(C) \otimes \mathfrak{g})^G$.

II: Cartan model (notable mentions)

There is a canonical choice of a quasi-inverse of p using a connection.
For any $\Omega \in (\Omega^2(C) \otimes \mathfrak{g})^G$ define

$$\Omega \colon T_{\mathrm{Cart}}(C) \times T_{\mathrm{Cart}}(C) \to T_{\mathrm{Cart}}(C)$$

by

$$\Omega(P \otimes X, Q \otimes Y) := e_i \vee P \vee Q \otimes \Omega^i_{\alpha\beta} \, \mathrm{i_a}(\mathrm{d}x^\alpha)(X) \wedge \mathrm{i_a}(\mathrm{d}x^\beta)(Y)$$

extend as coderivation of degree 0

$$\Omega \colon \mathrm{S}^\bullet(T^\bullet_{\mathrm{Cart}}(C)[1]) \to \mathrm{S}^\bullet(T^\bullet_{\mathrm{Cart}}(C)[1]).$$

Theorem

For a principal connection ω with curvature Ω the map

$$\mathrm{e}^\Omega \circ \mathrm{hor}_\omega$$

is a quasi-inverse of p.

Summary

We found DGLA morphisms

$$
\begin{array}{ccc}
(T^\bullet_{\mathfrak{g}}(M), \mathrm{d}^{\Pi}, [\,\cdot\,,\,\cdot\,]) & \xrightarrow{\;\mathcal{T}\;} & (T_{\mathrm{Tay}\,\mathfrak{g}}(C), \mathrm{d}^{\mathcal{T}(\Pi)}, [\,\cdot\,,\,\cdot\,]) \\
 & \searrow & \uparrow \simeq \\
 & & (T_{\mathrm{Cart}}(C), \partial, [\,\cdot\,,\,\cdot\,]) \\
 & & \downarrow \simeq \\
 & & (T^\bullet_{\mathrm{poly}}(M_{\mathrm{red}}), [\,\cdot\,,\,\cdot\,])
\end{array}
$$

for $M = C \times \mathfrak{g}^*$. BUT: not quite what we want! Desirable:

$$MW_{\mathrm{red}} \colon (T^\bullet_{\mathfrak{g}}(M), \lambda, \mathrm{d}^{-J}, [\,\cdot\,,\,\cdot\,]) \to (T^\bullet_{\mathrm{poly}}(M_{\mathrm{red}}), [\,\cdot\,,\,\cdot\,])$$

Aim

Find an explicit L_∞-quasi-inverse P of

$$\iota \colon (T_{\mathrm{Cart}}(C), \partial, [\,\cdot\,,\,\cdot\,]) \to (T_{\mathrm{Tay},\mathfrak{g}}(C), \mathrm{d}^{\mathcal{T}(\Pi)}, [\,\cdot\,,\,\cdot\,])$$

AND: use twisting.

L_∞ Deformation Retracts

Assume:

$$(A, \mathrm{d}_A, [\,\cdot\,,\,\cdot\,]_A) \underset{p}{\overset{i}{\rightleftarrows}} (B, \mathrm{d}_B, [\,\cdot\,,\,\cdot\,]_B) \circlearrowleft h$$

deformation retract of complexes + i DGLA map $\implies$ the transfered L_∞-structure on A is given by $(\mathrm{d}_A, [\,\cdot\,,\,\cdot\,]_A)$. Then:

Lemma

There is a rather explicit recursive formula for the Taylor coefficients of a quasi-inverse P with $P_1 = p$.

Not surprising: tensor trick!

The quasi-inverse

We use the twisting by $-\mathfrak{T}(\pi_{\mathrm{KKS}})$ of the constructed P in order to get:

$$P^{-\mathfrak{T}(\pi_{\mathrm{KKS}})} \colon (T_{\mathrm{Tay},\mathfrak{g}}(C), \mathrm{d}^{-\mathfrak{T}(J)}, [\,\cdot\,,\,\cdot\,]) \to (T_{\mathrm{Cart}}(C), P^{\mathfrak{T}(\pi_{\mathrm{KKS}})}(\mathfrak{T}(\lambda)), \partial^{\widetilde{\pi_{\mathrm{KKS}}}}, [\,\cdot\,,\,\cdot\,])$$

with

$$\widetilde{\pi_{\mathrm{KKS}}} := \sum \frac{1}{k!} P_k(\pi_{\mathrm{KKS}}^{\vee k}).$$

Lemma

We have

$$\widetilde{\pi_{\mathrm{KKS}}} = P^{\pi_{\mathrm{KKS}}}(\mathfrak{T}(\lambda)) = 0$$

hence

$$P^{-\mathfrak{T}(\pi_{\mathrm{KKS}})} \colon (T_{\mathrm{Tay},\mathfrak{g}}(C), \mathrm{d}^{-\mathfrak{T}(J)}, [\,\cdot\,,\,\cdot\,]) \to (T_{\mathrm{Cart}}(C), \partial, [\,\cdot\,,\,\cdot\,])$$

is L_∞-morphism.

Final step: Precise Statement

Theorem

Lie group action $\Phi\colon G \times M \to M$ + $J\colon M \to \mathfrak{g}^*$ equivariant with $0 \in \mathfrak{g}^*$ regular value. If G acts properly in a neighbourhood of $C = J^{-1}(0)$, then there exists an open subset $U \subset C \times \mathfrak{g}^*$ containing $C \times \{0\}$, such that

1. *U is diffeomorphic to an open set in M containing C.*
2. *J is given by the projection to $\mathfrak{g}^*$*

With this:

Theorem

Lie group action $\Phi\colon G \times M \to M$ + $J\colon M \to \mathfrak{g}^*$ equivariant with $0 \in \mathfrak{g}^*$ regular value. If G acts properly around $C = J^{-1}(0)$ and free on C, then there is a L_∞-morphism

$$MW_{\mathrm{red}}\colon (T^\bullet_{\mathfrak{g}}(M)[[\hbar]], \hbar\lambda, \mathrm{d}^{-J}, [\,\cdot\,,\,\cdot\,]) \to (T^\bullet_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]], [\,\cdot\,,\,\cdot\,])$$

inducing the Marsden-Weinstein/BRST reduction on the level of Maurer-Cartan elements.

And now?

Same game for

$$D^k_{\mathfrak{g}}(M) = \bigoplus_{2i+j=k} (\mathrm{S}^i\mathfrak{g}^* \otimes D^j_{\mathrm{poly}}(M))^G.$$

Maurer-Cartan elements of

$$(D^\bullet_{\mathfrak{g}}(M)[[\hbar]], \hbar\lambda, \partial_G^{-J}, [\,\cdot\,,\,\cdot\,])$$

are *equivariant star products.*

Conjecture

There exists a L_∞-morphism

$$MW_{\mathrm{red}} \colon (D^\bullet_{\mathfrak{g}}(M)[[\hbar]], \hbar\lambda, \partial_G^{-J}, [\,\cdot\,,\,\cdot\,]) \to (D_{\mathrm{poly}}(M_{\mathrm{red}}), \partial, [\,\cdot\,,\,\cdot\,])$$

Inducing (up to equivalence) the BRST-reduction of star products on the level of Maurer-Cartan elements.

Thank you!