Lie groups of Poisson diffeomorphisms

Friday Fish Seminar

Wilmer Smilde

Based on my master's thesis, supervised by Ioan Mărcuț

13-11-2020

Contents

Overview and the problem

Algebraic Aspects

Global

- $\text{Diff}(M, \pi) = \{\text{Poisson diffeomorphisms}\}$
- $\text{Fol}(M, \pi) = \{\text{Poisson diffeomorphisms preserving each symplectic leaf}\}$
- $\text{Ham}_{\text{loc}}(M, \pi) = \{\text{Time-1 flows of (time-dep.) locally Hamiltonian vector fields}\}$
- $\text{Ham}(M, \pi) = \{\text{Time-1 flows of (time-dependent) Hamiltonian vector fields}\}$

Infinitesimal

- $\mathfrak{X}(M, \pi) = \{\text{Poisson vector fields}\}$
- $\mathfrak{fol}(M, \pi) = \{\text{Poisson vf's tangent to the symplectic foliation}\}$
- $\mathfrak{ham}_{\text{loc}}(M, \pi) = \{\pi^\sharp(\alpha) : \alpha \in \Omega^1(M) \textit{ closed}\}$
- $\mathfrak{ham}(M, \pi) = \{\text{Hamiltonian vector fields}\} = \{\pi^\sharp(df) : f \in C^\infty(M)\}$

Examples

Example

Symplectic manifold (M, ω)

$$\mathrm{Ham}(M,\omega) \subset \mathrm{Ham}_{\mathrm{loc}}(M,\omega) = \mathrm{Fol}(M,\omega) = \mathrm{Diff}_0(M,\omega)$$

The difference $\mathrm{Ham}_{loc} / \mathrm{Ham}$ is described by (a quotient of) $H^1(M)$.

Example

A manifold M with $\pi = 0$

$$\{\mathrm{id}\} = \mathrm{Ham}(M,0) = \mathrm{Ham}_{\mathrm{loc}}(M,0) = \mathrm{Fol}(M,0) \subset \mathrm{Diff}(M,0) = \mathrm{Diff}(M)$$

The problem

Problem
Can we make sense of $\mathrm{Diff}(M, \pi)$ *as a Lie group with Lie algebra* $\mathfrak{X}(M, \pi)$*? What about its subgroups?*

First, understand manifold structure on $\mathrm{Diff}(M)$

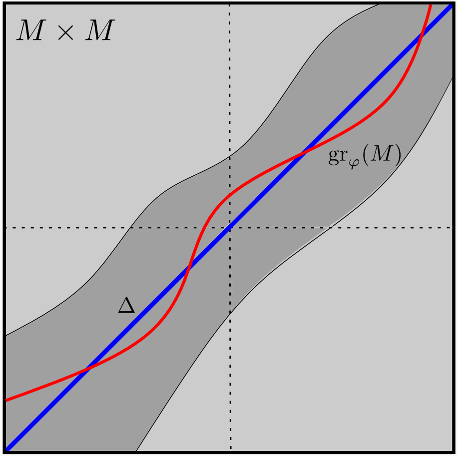
The local nature of the diffeomorphism group

Manifold charts for Diff(M)

- Any map $\varphi : M \to M$ induces a graph $\mathrm{gr}_\varphi : M \to M \times M$
- In a tubular neighbourhood of Δ,

$$\left\{\begin{matrix}\text{diffeomorphisms} \\ C^1\text{-close to id}\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{sections of } N\Delta \\ C^1\text{-close to } 0\end{matrix}\right\}$$

- $N\Delta \cong TM$
- Diff(M) is a Lie group with Lie algebra $(\mathfrak{X}(M), [\cdot,\cdot])$

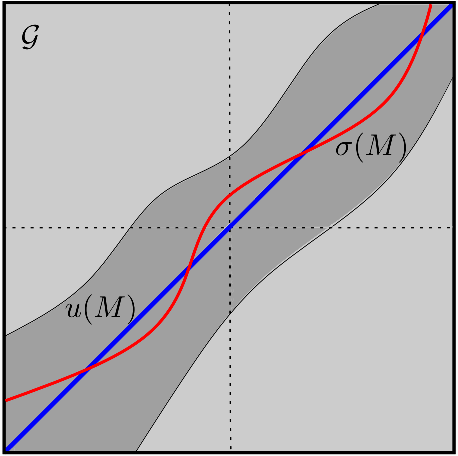


Manifold charts for $\mathrm{Bis}(\mathcal{G})$

- A bisection of $\mathcal{G} \rightrightarrows M$ is $\sigma : M \to \mathcal{G}$ s.t. $s \circ \sigma = \mathrm{id}_M$ and $t \circ \sigma$ is a diffeomorphism
- In a tubular neighbourhood of $M \subset \mathcal{G}$,

$$\left\{\begin{matrix}\text{bisections} \\ C^1\text{-close to id}\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{sections of } NM \\ C^1\text{-close to } 0\end{matrix}\right\}$$

- $NM \cong \mathcal{A} = \mathrm{Lie}(\mathcal{G})$
- $\mathrm{Bis}(\mathcal{G})$ is a Lie group with Lie algebra $(\Gamma(\mathcal{A}), [\cdot,\cdot])$



Local nature of the symplectomorphism group

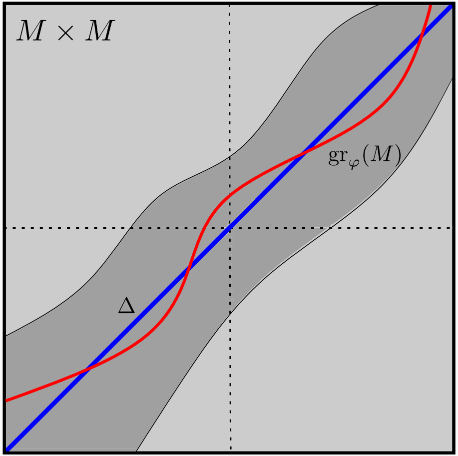
Let (M, ω) be a symplectic manifold

- $\varphi \in \mathrm{Diff}(M)$ is symplectic if and only if $\mathrm{gr}_\varphi \subset (M \times M, \omega \times (-\omega))$ is Lagrangian
- Lagrangian tubular neighbourhood theorem:

$$(T^*M, \omega_{\mathrm{can}}) \xrightarrow[\cong]{\text{local}} (M \times M, \omega \times (-\omega))$$

- In this tubular neighbourhood of Δ,

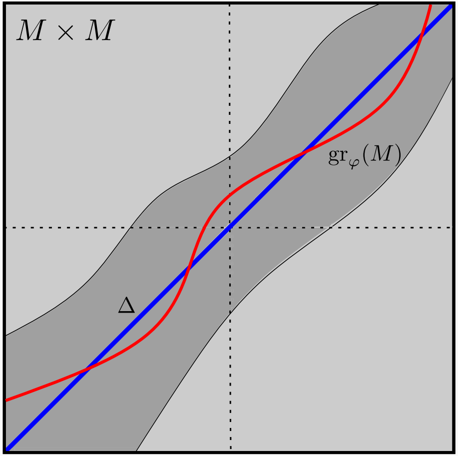
$$\left\{ \begin{array}{c} \text{symplecto-} \\ \text{morphisms} \\ C^1\text{-close to id} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textbf{closed} \text{ sections} \\ \text{of } T^*M \\ C^1\text{-close to } 0 \end{array} \right\}$$



Local nature of the Poisson diffeomorphism group

Let (M, π) be a Poisson manifold

- $\varphi \in \mathrm{Diff}(M)$ is Poisson if and only if $\mathrm{gr}_\varphi \subset (M \times M, \pi \times (-\pi))$ is coisotropic
- Local nature is of $\mathrm{Diff}(M, \pi)$ is about coisotropic deformations of Δ!
- Problem: rarely, the deformations of $\Delta \subset M \times M$ are easily described



Strategy

Observations: let $(\mathcal{G}, \Pi) \rightrightarrows M$ be a Poisson groupoid

- Coisotropic bisections $\mathrm{Bis}(\mathcal{G}, \Pi) \subset \mathrm{Bis}(\mathcal{G})$
- Induced homomorphism
$$\mathrm{Bis}(\mathcal{G}, \Pi) \to \mathrm{Diff}(M, \pi)$$
- Local nature of $\mathrm{Bis}(\mathcal{G}, \Pi)$ is about coisotropic deformations of $M \subset \mathcal{G}$

Strategy
Given a Poisson manifold (M, π), find a Poisson groupoid $(\mathcal{G}, \Pi) \rightrightarrows (M, \pi)$ so that

1. $\mathrm{Bis}(\mathcal{G}, \Pi)$ *is an interesting group;*
2. $\mathrm{Bis}(\mathcal{G}, \Pi)$ *is a Lie group.*

Linearization

The linearization problem

- $\mathcal{A} \to L$ a Lie algebroid induces a linear Poisson structure on $\mathcal{A}^*$

Let $L \subset (M, \pi)$ be a coisotropic submanifold

- The conormal bundle $N^*L \to L$ is a subalgebroid of $\pi^\sharp : T^*M \to M$
- Induced linear Poisson structure π_{lin} on NL is the *linearization* of π around L
- Upshot: π is linear $\implies$ coisotropic deformation space is linear
- If π is linear, then L is Lagrangian

$$(TL)^{\perp_\pi} = TL \cap \mathrm{Im}\pi^\sharp$$

Problem
Let $L \subset (M, \pi)$ be a Lagrangian submanifold. When is π_{lin} locally isomorphic to π?

Proposition
If $(\mathcal{G}, \Pi) \rightrightarrows M$ is a Poisson groupoid. Then M is a Lagrangian submanifold of $\mathcal{G}$.

More on linear Poisson structures

Example: $\mathcal{A} = TL \to L$

- On T^*L, canonical one-form $\lambda_{\text{can}} \in \Omega^1(T^*L)$

$$(\lambda_{\text{can}})_\alpha(v) = \alpha(T_\alpha p(v))$$

- The symplectic form $\omega_{\text{can}} = d\lambda_{\text{can}}$ induces the linear Poisson structure on T^*L
- The image of a section $\alpha : L \to T^*L$ is Lagrangian if and only if

$$\alpha^*\omega_{\text{can}} = d\alpha = 0$$

- Lagrangian neighbourhood theorem: symplectic structures are linearizable around Lagrangian submanifolds

More on linear Poisson structures *(de Leon, Marrero, Martinez '04)*

Let $\mathcal{A} \to L$ be a Lie algebroid, and let $p : \mathcal{A}^* \to L$ be the projection

- The algebroid $p^!\mathcal{A} \to \mathcal{A}^*$ has a canonical one form $\lambda_{\mathrm{can}} \in \Omega^1(p^!\mathcal{A})$

$$(\lambda_{\mathrm{can}})_\alpha(v) = \alpha(p_!(v))$$

- The two-form $\omega_{\mathrm{can}} = d\lambda_{\mathrm{can}} \in \Omega^2(p^!\mathcal{A})$ is symplectic and induces the linear Poisson structure

$$\begin{array}{ccc} (p^!\mathcal{A})^* & \xleftarrow{\omega^\flat_{\mathrm{can}}} & p^!\mathcal{A} \\ {\scriptstyle \rho^*}\uparrow & & \downarrow{\scriptstyle \rho} \\ T^*\mathcal{A} & \xrightarrow{\pi_{\mathrm{lin}}} & T\mathcal{A} \end{array}$$

- The image of a section $\alpha : L \to \mathcal{A}^*$ is Lagrangian if and only if

$$\alpha^*\omega_{\mathrm{can}} = d\alpha^*\lambda_{\mathrm{can}} = d\alpha = 0$$

- Coisotropic deformations of $L \subset \mathcal{A}^*$ are Lagrangian deformations

A Lagrangian neighbourhood theorem

Definition
Let $(\mathcal{A}, \omega) \to M$ be a symplectic Lie algebroid. A *Lagrangian transversal* is a submanifold $i : L \to M$ that is transverse to $\mathcal{A}$ and $(i^!\mathcal{A})^{\perp_\omega} = i^!\mathcal{A}$.

Theorem
Let $(\mathcal{A}, \omega) \to M$ be a symplectic Lie algebroid. Let $i : L \to M$ be a Lagrangian transversal. Then around L there is a local symplectomorphism

$$(\mathcal{A} \to M, \omega) \xrightarrow[\cong]{\text{local}} (p^! i^! \mathcal{A} \to (i^!\mathcal{A})^*, \omega_{\text{can}})$$

that restricts the identity on L.

Key ingredients for the proof

- Splitting theorem for Lie algebroids $p^! i^! \mathcal{A} \cong \mathcal{A}|_U$ (Bursztyn-Lima-Meinrecken '17)
- A Moser-Weinstein stability result (also announced by Sjamaar in June '20)

Examples

$(\mathcal{A}, \omega) \to M$ symplectic Lie algeboid, $i : L \to M$ a Lagrangian transversal

- $\mathcal{A} = TM$
- $(\mathcal{F}, \omega) \subset TM$ a symplectic foliation
- $\mathcal{A} = {}^bT_Z M$, the log-tangent bundle, and $L \pitchfork Z$
- b^k-tangent bundle, elliptic tangent bundle etc...

Applications

Application: Lagrangian bisections (Ping Xu '97, Rybicki '01)

Let $(\mathcal{G}, \Omega) \rightrightarrows (M, \pi)$ be a symplectic groupoid

- Lagrangian bisections $\sigma : M \to \mathcal{G}$ form a subgroup $\mathrm{Bis}(\mathcal{G}, \Omega) \subset \mathrm{Bis}(\mathcal{G})$
- Lagrangian tubular neighbourhood theorem:

$$(T^*M, \omega_{\mathrm{can}}) \supset U \xrightarrow{\Phi} V \subset (\mathcal{G}, \Omega)$$

- In this tubular neighbourhood of Δ,

$$\left\{ \begin{matrix} \text{Lagrangian} \\ \text{bisections} \\ C^1\text{-close to id} \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \textbf{closed} \text{ sections} \\ \text{of } T^*M \\ C^1\text{-close to } 0 \end{matrix} \right\}$$

- $\mathrm{Bis}(\mathcal{G}, \Omega)$ is a Lie group with Lie algebra $\left(\Omega^1_{\mathrm{cl}}(M), [\cdot,\cdot]_\pi\right)$

$$\begin{array}{ccc} \mathrm{Bis}_{\mathrm{exact}}(\mathcal{G}, \Omega) & \hookrightarrow & \mathrm{Bis}(\mathcal{G}, \Omega) \\ \downarrow & & \downarrow \\ \mathrm{Ham}(M, \pi) & \hookrightarrow & \mathrm{Ham}_{\mathrm{loc}}(M, \pi) \end{array} \qquad \begin{array}{ccc} \Omega^1_{\mathrm{exact}}(M) & \hookrightarrow & \Omega^1_{\mathrm{cl}}(M) \\ \downarrow \pi^\sharp & & \downarrow \pi^\sharp \\ \mathfrak{ham}(M, \pi) & \hookrightarrow & \mathfrak{ham}_{\mathrm{loc}}(M, \pi) \end{array}$$

Application: Log-symplectic structures

Definition
A Poisson structure π on M is log-symplectic if $\wedge^n \pi : M \to \wedge^{2n} TM$ is transverse to the zero section.

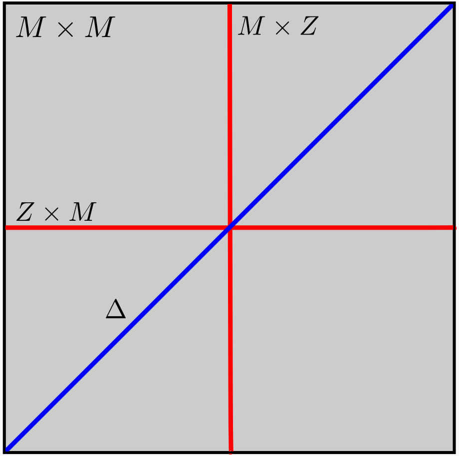
- π is symplectic away from a hypersurface Z
- on Z, it induces a corank-one Poisson structure
- Equivalently, π comes from a symplectic form on the log-tangent bundle ${}^bTM$

$$
\begin{array}{ccc}
{}^bT^*M & \xleftarrow{\omega^\flat} & {}^bTM \\
\uparrow & & \downarrow \\
T^*M & \xrightarrow{\pi^\sharp} & TM
\end{array}
$$

Application: Log-symplectic manifolds

Let (M, π) be a log-symplectic manifold

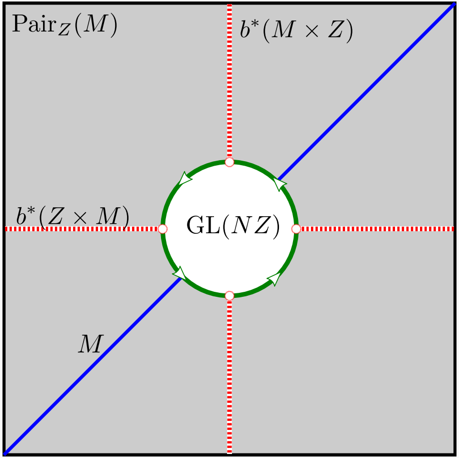
- Problem: $\pi \times (-\pi)$ is **not** linearizable around Δ



Application: Log-symplectic manifolds

Let (M, π) be a log-symplectic manifold

- Solution: Blow-up $Z \times Z$ in $M \times M$ to $\mathrm{Pair}_Z(M)$ (Gualtieri-Li)
- $(\mathrm{Pair}_Z(M), \Pi) \rightrightarrows (M, \pi)$ is a Poisson groupoid, and Π is log-symplectic
- M is transverse to degeneracy locus of Π
- Apply Lagrangian neighbourhood theorem to obtain a manifold structure on $\mathrm{Bis}(\mathrm{Pair}_Z(M), \Pi)$



Application: Log-symplectic manifolds

Theorem

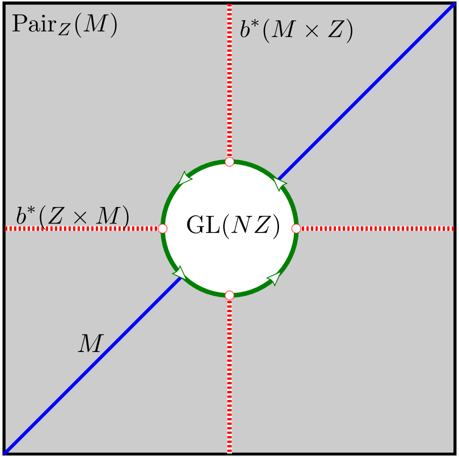
Let M be a manifold, $Z \subset M$ a hypersurface. Then

$$\mathrm{Bis}(\mathrm{Pair}_Z(M)) \cong \mathrm{Diff}(M, Z)$$

is a Lie group with Lie algebra $\Gamma({}^b T_Z M)$. Let (M, π) be a log-symplectic manifold. Then

$$\mathrm{Bis}(\mathrm{Pair}_Z(M), \Pi) \cong \mathrm{Diff}(M, \pi)$$

is a Lie group with Lie algebra ${}^b\Omega^1_{\mathrm{cl}}(M) \cong \mathfrak{X}(M, \pi)$.



Application: Log-symplectic manifolds

What about foliated diffeomorphisms?

- Infinitesimally:
$$ {}^b\Omega^1_{\mathrm{cl}}(M) \cong \mathfrak{X}(M,\pi), \quad \Omega^1_{\mathrm{cl}}(M) \cong \mathfrak{fol}(M,\pi) $$
- Under additional assumptions, $(\mathrm{Pair}_\pi(M), \Omega)$ symplectic groupoid obtained by a blowup in $\mathrm{Pair}_Z(M)$ (Gualtieri-Li)
- $\mathrm{Fol}(M,\pi) \cong \mathrm{Bis}(\mathrm{Pair}_\pi(M), \Omega)$

Application: Log-symplectic manifolds

$$\begin{array}{ccccccc}
\mathrm{Bis}_{\mathrm{ex}}(\mathrm{Pair}_\pi(M),\Omega) & \longrightarrow & \mathrm{Bis}_0(\mathrm{Pair}_\pi(M),\Omega) & \longrightarrow & \mathrm{Bis}(\mathrm{Pair}_\pi(M),\Omega) & \longrightarrow & \mathrm{Bis}(\mathrm{Pair}_Z(M),\Pi) \\
\downarrow \cong & & \downarrow \cong & & \downarrow {\color{red}\cong} & & \downarrow \cong \\
\mathrm{Ham}(M,\pi) & \longrightarrow & \mathrm{Ham}_{\mathrm{loc}}(M,\pi) & \longrightarrow & \mathrm{Fol}(M,\pi) & \longrightarrow & \mathrm{Diff}(M,\pi)
\end{array}$$

Other Lie groups

- Scattering manifolds (M, Z, ω)

$$\mathrm{Ham}(M,\omega) \subsetneq \mathrm{Ham}_{\mathrm{loc}}(M,\omega) \subsetneq \mathrm{Fol}_0(M,\omega) \subsetneq \mathrm{Diff}_0(M,\omega)$$

- Poisson structures coming from cosymplectic structures
- Poisson manifolds of strong proper type (an application of a recent linearization result by Aldo Witte)