Utrecht University

Haefliger's differentiable cohomology

Variations on variations on a theorem of Van Est

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Utrecht University

What are we going to do today?

Some context

Based on Haefliger's *Differential cohomology*, 1976, Varenna.

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- Geometric structures on manifolds M come with invariants in the cohomology ring of M.
- Those are organised in characteristic map from some kind of "universal space".
- Haefliger's cohomology:
  - arose in the development of characteristic classes for foliations $\mathcal{F}$ on manifolds M;
  - was built having in mind an analogy with flat principal bundles and their characteristic classes.

Characteristic classes of principal bundles

The classical picture for G-principal bundles $P \to M$

$$\begin{array}{ccc} \mathrm{Inv}(\mathfrak{g}) & \xrightarrow{\kappa^P_{CW}} & H^*(M) \\ {\scriptstyle \kappa^{\mathrm{univ}}}\downarrow & \nearrow{\scriptstyle \kappa^P_{\mathrm{abs}}} & \\ H^*(BG) & & \end{array}$$

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becomes the following if P carries a flat connection ω:

$$\begin{array}{ccc} H^*(\mathfrak{g}, K) & \xrightarrow{\kappa^P_{\omega}} & H^*(M) \\ {\scriptstyle \kappa^{\mathrm{univ}}}\downarrow & \nearrow_{\kappa^P_{\mathrm{abs}}} & \\ H^*(G^\delta) & & \end{array}$$

plus the Van Est isomorphism:

$$H^*_d(G) \xrightarrow{\cong} H^*(\mathfrak{g}, K).$$

The cocycle approach to foliations

Haefliger's groupoid

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I.e.: *foliations are cocycles valued in the groupoid* Γ^q.

Characteristic classes of foliations

The theory produces a diagram of the form

$$\begin{array}{ccc} H^*(\mathfrak{a}_q, O(q)) & \xrightarrow{\kappa^{\mathcal{F}}} & H^*(M) \\ {\scriptstyle \kappa^{\mathrm{univ}}}\downarrow & \nearrow_{\kappa^{\mathcal{F}}_{\mathrm{abs}}} & \\ H^*(B\Gamma^q) & & \end{array}$$

$\mathfrak{a}_q$ is the Lie algebra of formal vector fields.

- $\kappa^{\mathcal{F}}$ is a "geometric map", defined on a computable Lie algebra cohomology $H^*(\mathfrak{a}_q, O(q))$ (as in "flat" Chern-Weil theory).

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    - can be described via a "de Rham-like" approach (Bott-Shulman complex for the groupoid Γ^q)
    - or through sheaf cohomology plus bar-type resolutions (i.e. group-like cochains, on Γ^q, as for discrete groups).

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one can ask oneself

- whether there is a "differentiable complex" for Γ^q, coming with a Van Est-like isomorphism to $H^*(\mathfrak{a}_q, O(q))$;
- whether there is some "flat connection" around inducing the "geometric" characteristic map.

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- We are going to clarify Haefliger's construction and provide the conceptual framework where it belongs.

Getting started: groupoids, cocycles, geometric structures

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Lie groupoid: $\Gamma, \mathbf{X}$ are manifolds, all the operations and maps are smooth, s, t are submersions.

Étale Lie groupoids

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A Lie groupoid $s, t : \Gamma \rightrightarrows \mathbf{X}$ is étale if s, t are étale maps.

E.g.: the groupoid

$$\Gamma^{\mathbf{X}} := \mathrm{Germ}(\mathrm{Diff}_{\mathrm{loc}}(\mathbf{X})) \rightrightarrows \mathbf{X},$$

with the germ topology, is étale.

Effectivness

Bisections: sections $\sigma : \mathbf{X} \to \Gamma$, $t \circ \sigma$ is a diffeomorphism.

Γ étale: $g \in \Gamma$ corresponds to a unique bisection.

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Effectivness

Γ is effective if the functor $\Gamma \to \Gamma^{\mathbf{X}}$ is faithful.

Pseudogroups

Theorem (Haefliger)

Effective étale Lie groupoids over $\mathbf{X}$ are in $1:1$ correspondence with pseudogroups $\mathbf{\Gamma}$ over $\mathbf{X}$, i.e. subsheaves of $\mathrm{Diff}_{\mathrm{loc}}(\mathbf{X})$ which are closed w.r.t. composition, inversion and have $id_{\mathbf{X}}$ as a section.

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- $\mathbf{\Gamma} \to \Gamma := \mathrm{Germ}(\mathbf{\Gamma})$.
- $\Gamma \to \mathbf{\Gamma} := t(\mathrm{Bis}_{\mathrm{loc}}(\Gamma))$.

Cocycles

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- an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M;

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  - $\gamma_{ik}(x) = \gamma_{jk} \circ \gamma_{ij}(x)$ holds for all i, j, k, $x \in U_i \cap U_j \cap U_k$.

Haefliger's approach to foliations

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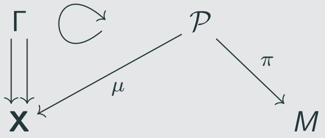
Γ: additional transverse structure, whose local symmetries are controlled by $\mathbf{\Gamma}$. One gets Γ-foliations and Haefliger Γ-structures.

One declares two Γ-cocycles indexed by I and J to be equivalent if they are part of a larger cocycle indexed by $I \coprod J$.

Cocycles and principal bundles

Equivalence classes of cocycles in $\mathbf{\Gamma}$ correspond to isomorphism classes of principal Γ-bundles:



The abstract map: the classifying space and its cohomology

Haefliger's classifying space

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- $H^*(BG^\delta) \cong H^*(G^\delta)$, the group cohomology of G.

A model for $H^*(B\Gamma)$: groupoid cohomology of Γ

- Continuous Γ-sheaf $\mathcal{S}$: sheaf over **X** & right Γ-space.

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- Γ-sheaves form an abelian category $\mathrm{Ab}(\Gamma)$.

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The k-groupoid cohomology $H^k(\Gamma, \mathcal{S})$: k-th right derived functor of the functor of invariant sections

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Needed: injective resolutions by Γ-sheaves.

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Theorem (Moerdijk '98)

$H^*(\Gamma, \mathcal{S}) \cong H^*(B\Gamma, \hat{\mathcal{S}})$, for a suitable induced sheaf $\hat{\mathcal{S}}$.

The constant sheaf $\mathbb{R}$ is a Γ-sheaf; $\hat{\mathbb{R}} = \mathbb{R}$.

A model for $H^*(B\Gamma)$: groupoid cohomology of Γ

Nerve of Γ: a simplicial manifold associated to Γ.

$\Gamma^{(p)}$: the space of composable p-strings.

Face maps: $d_i : \Gamma^{(p)} \to \Gamma^{(p-1)}$.

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$C^p(\Gamma, \mathcal{S})$: sections of $t^*\mathcal{S}$, $t : \Gamma^{(p)} \to \mathbf{X}$ target of the first element.

Induced groupoid differential:

$$\delta : C^p(\Gamma, \mathcal{S}) \to C^{p+1}(\Gamma, \mathcal{S}),$$

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Theorem (Haeliger '79)

There is a canonical map $H^*_{\mathrm{dR}}(\Gamma) \to H^*(\Gamma, \mathbb{R})$ which is an isomorphism when Γ is Hausdorff.

A differentiable complex for Γ

The "soft" topology

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Haefliger's approach: smooth cochains on Γ^q are smooth w.r.t. soft topology and valued in smooth representations.

A cleaner approach: jet groupoids and Lie pseudogroups

We change the groupoid, not the topology.

k-th jet groupoid: $J^k\Gamma = \{j^k_x\varphi : \ x \in \mathbf{X}, \varphi \in \mathbf{\Gamma}\}$

Lie pseudogroups

If the tower

$$\cdots \to J^k\Gamma \to J^{k-1}\Gamma \to \cdots \to J^0\Gamma \rightrightarrows \mathbf{X}$$

is a tower of surjective submersions between smooth manifolds and $J^\infty\Gamma \cong \varprojlim J^k\Gamma$, $\mathbf{\Gamma}$ is called Lie pseudogroup.

The smooth structure of $J^\infty\Gamma$

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E.g.: $\Omega^*(J^\infty\Gamma) := \varinjlim \Omega^*(J^k\Gamma)$ → classical Cartan calculus.

The natural map $j : \Gamma \cong \mathrm{Germ}(\mathbf{\Gamma}) \to J^\infty\Gamma$ *is smooth.*

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Differentiable cohomology

The differentiable cohomology of Γ is the cohomology of the simple complex associated to the double complex (?)

$$C^p_{\mathrm{diff}}(\Gamma, \Lambda^q T^*\mathbf{X}) \hookrightarrow C^p(\Gamma, \Omega_{\mathbf{X}}) = \Omega^q(\Gamma^{(p)})$$

The wish list

$\Sigma \rightrightarrows \mathbf{X}$, Lie groupoid. We want to equip $C_d^p(\Sigma, \Lambda^q T^*\mathbf{X})$ with

- horizontal differentials

$$\delta^q : C_d^p(\Sigma, \Lambda^q T^*\mathbf{X}) \to C_d^{p+1}(\Sigma, \Lambda^q T^*\mathbf{X}).$$

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  - δ^* and d^* are compatible (i.e. $\to$ double complex).

Differentials: representations and connections

Horizontal differentials: representations

$\delta^q : C_d^p(\Sigma, \Lambda^q T^*\mathbf{X}) \to C_d^{p+1}(\Sigma, \Lambda^q T^*\mathbf{X})$ is equivalent to a representation of Σ on $\Lambda^q T^*X$, $q \geq 1$.

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Vertical differentials: connections

$d^p : C^p_d(\Sigma, \Lambda^q T^*\mathbf{X}) \to C^p_d(\Sigma, \Lambda^{q+1} T^*\mathbf{X})$ is equivalent to a flat Ehresmann connection of $\Sigma^{(p)}$, $p \geq 1$.

True replacing $t : \Sigma^{(p)} \to \mathbf{X}$ with any submersion $P \to X$.

Leibniz identities: it is simpler than it looks

Only one representation; only one connection

The Leibniz identities imply:

- the representation on $\Lambda^q T^*X$ is the induced diagonal action of the action on T^*X;
- $H^p = \{(v_1, \ldots, v_p) \in T\Sigma^{(p)} : v_1, \ldots v_p \in H^1\}$.

Hence: *we need one representation on* TX *and one flat connection* $\mathcal{C} := H^1$ *on* Σ*!*

Compatibility: one multiplicative connection

Compatibility condition

$(C^p(\Sigma, \Lambda^q T^*X), \delta, d_{\mathcal{C}})$ is a double complex iff $\mathcal{C}$ is multiplicative.

$\mathcal{C} := H^1$ induces a "quasi-action":

$$a_g^{\mathcal{C}} : T_y\mathbf{X} \to T_x\mathbf{X}, \quad a_g(v) = ds(\mathrm{hor}_g^{\mathcal{C}}(v))$$

Multiplicativity implies that this is the representation from δ.

Conclusion: flat Cartan groupoids

Cartan groupoid: a pair $(\Sigma, \mathcal{C})$ s.t.

- $\Sigma \rightrightarrows \mathbf{X}$ is a Lie groupoid;
- $\mathcal{C}$ is a multiplicative Ehresmann connection on Σ.

It is called flat if $\mathcal{C}$ is a flat connection.

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Theorem (A., Crainic)

A Lie groupoid has a Haefliger bicomplex

$$(C^p_d(\Sigma, \Lambda^q T^*\mathbf{X}), \delta, d)$$

as above iff it is a flat Cartan groupoid.

Its cohomology $H^*_{\mathrm{Hae}}(\Sigma, \mathcal{C})$ is called Haefliger cohomology.

Flat Cartan groupoids: the example we care about

Theorem

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- Cartan tower:

$$\cdots \to (J^k\Gamma, \mathcal{C}^k) \overset{\pi^{k,k-1}}{\to} (J^{k-1}\Gamma, \mathcal{C}^{k-1}) \to \dots$$

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  - $d\pi^{k,k-1}(\mathcal{C}^k \cap \ker(ds)) = 0$;

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  - $[d\pi^{k,k-1}(\mathcal{C}^k), d\pi^{k,k-1}(\mathcal{C}^k)] \subset \mathcal{C}^{k-1}$.

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Differentiable cohomology of Γ:

$$H^*_{\mathrm{diff}}(\Gamma) := H_{\mathrm{Hae}}(J^\infty\Gamma, \mathcal{C}^\infty).$$

We have the map

$$j^* : H^*_{\mathrm{Hae}}(J^\infty\Gamma, \mathcal{C}^\infty) \to H^*(\Gamma, \mathbb{R}), \quad j : \Gamma \to J^\infty\Gamma$$

Van Est maps

Proper actions: a general Van Est map

Let $\mu : P \to \mathbf{X}$ be Σ-space.

$(\Sigma \ltimes P, pr_1^{-1}(\mathcal{C}))$ is flat Cartan.

$\Omega^*(P)^\Sigma$: subcomplex of Σ-invariant forms.

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Theorem (A., Crainic)

If $(\Sigma, \mathcal{C})$ acts properly on P then there is a natural map

$$VE_P : H^*_{\mathrm{Hae}}(\Sigma, \mathcal{C}) \to H^*_{\Sigma}(P)$$

Moreover, if μ is submersive with contractible fibers then VE_P is an isomorphism.

A glimpse at the infinitesimal picture: extended isotropy

$A = \mathrm{Lie}(\Sigma)$, the Lie algebroid of Σ.

$\rho : A \to T\mathbf{X}$ anchor map; $[\ ,\ ] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ Lie bracket.

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Lemma

$\mathcal{C}$ induces a Lie bracket $\{\ ,\ \}_{\mathrm{pt}}$ on $\Gamma(A)$ such that $(A, \{\ ,\ \}_{\mathrm{pt}})$ is a Lie algebra bundle. The isotropy Lie algebra $\mathfrak{g}_x = \ker(\rho)_x$ is a subalgebra of $(A_x, \{\ ,\ \}_{\mathrm{pt}})$.

Notation: $(\mathfrak{a}_x(A), \{\ ,\ \}_{\mathrm{pt}})$, the extended isotropy Lie algebra.

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- Γ-vector field: X vector field on $\mathbf{X}$ s.t. flow lies in $\mathbf{\Gamma}$.

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$$\{j_x^\infty X, j_x^\infty Y\}_{\mathrm{pt}} = j_x^\infty [X, Y].$$

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For Γ^q the extended isotropy at 0, denoted by $\mathfrak{a}_q$, is the Gelfand-Fuchs algebra of formal vector fields.

Van Est map II

Theorem (A., Crainic)

If Σ is transitive, there is a natural isomorphism

$$VE_x : H^*_{\mathrm{Hae}}(\Sigma, \mathcal{C}) \xrightarrow{\cong} H^*(\mathfrak{a}_x(A), K)$$

where K is a subgroup of Σ_x such that Σ_x/K is contractible.

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- $\mathcal{X}(s^{-1}(x))^{\Sigma} \xrightarrow{\cong} \mathfrak{a}_x(A)$.
- Pass to K-basic cochains.

Haefliger isomorphism

When $\Sigma = J^\infty \Gamma^q$, one gets

$$H^*_{\mathrm{Hae}}(J^\infty \Gamma^q, \mathcal{C}^\infty) \overset{\cong}{\to} H^*(\mathfrak{a}_q, O(q))$$

$O(q) \subset (J^\infty \Gamma^q)_x$ as infinite jets of orthogonal maps.

This is the Van Est-like isomorphism proven by Haefliger.

Formal structures: geometric map

A general "geometric" characteristic map

Let $\pi : P \to M$ be principal Σ-bundle, $\mathcal{C}_P \subset TP$.

- $a : \Sigma \times P \to P$ multiplicative w.r.t. $\mathcal{C}_P$: $\mathcal{C} \cdot \mathcal{C}_P \subset \mathcal{C}_P$.

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Theorem (A., Crainic)

$(P, \mathcal{C}_P)$ flat principal $(\Sigma, \mathcal{C})$-bundle. There is a natural map

$$\kappa^P_{\mathrm{Hae}} : H^*_{\mathrm{Hae}}(\Sigma, \mathcal{C}) \to H^*(M)$$

Formal Haefliger structures

Let $(\Sigma, \mathcal{C}) = (J^\infty\Gamma, \mathcal{C}^\infty)$.

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Formal Haefliger structures

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- Flat principal $(\Sigma, \mathcal{C})$-bundles $\leftrightarrow$ formal Haefliger Γ-structure.
- Haefliger Γ-structure $\rightarrow$ formal integrable Haefliger Γ-structure
- Not all formal structures are integrable.

“Geometric” characteristic map

Theorem (A., Crainic)

For (integrable formal) Haefliger Γ-structures $P \to M$,

$$\begin{array}{ccc} H^*_{\mathrm{diff}}(\Gamma) & \xrightarrow{\kappa^P_{\mathrm{Hae}}} & H^*(M) \\ {\scriptstyle j^*}\downarrow & \nearrow{\scriptstyle \kappa^{\mathcal{P}}_{\mathrm{abs}}} & \\ H^*(B\Gamma) & & \end{array}$$

is commutative.

k^P_{Hae} *is defined regardless of integrability!*

Combining with Van Est isomorphism:

$$\kappa^P_{\mathrm{geo}} : H^*(\mathfrak{a}_x(A), K) \to H^*(M)$$

Thank you!